Renormalization of the chromomagnetic operator in HQET and the B*-B mass splitting

ALPHA Collaboration

SFB/TR 9

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DESY, Zeuthen

Trinity College, Dublin, April 2007

based on

Non-perturbative renormalization of the chromo-magnetic operator in Heavy Quark Effective Theory and the $B^* - B$ mass splitting.
D. Guazzini, H. Meyer & R.S., May 2007

numbers presented here are preliminary (may change within the errors)

HQET

QCD:

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2g_0^2}\,\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\} + \sum_f \overline{\psi}_f[D_\mu\gamma_\mu + m_f]\psi_f$$

HQET

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$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2g_0^2}\,\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\} + \sum_f \overline{\psi}_f[D_\mu\gamma_\mu + m_f]\psi_f$$

HQET: in the rest frame of a B-meson

$$\begin{aligned}
\overline{\psi}_{\mathrm{b}}[D_\mu\gamma_\mu + m_{\mathrm{b}}]\psi_{\mathrm{b}} &\rightarrow \mathcal{L}_{\mathrm{stat}} + \mathcal{L}^{(1)} + \mathrm{O}(1/m_{\mathrm{b}}^2)\,, \\
\mathcal{L}_{\mathrm{stat}} &= \overline{\psi}_{\mathrm{h}}[D_0 + m_{\mathrm{b}}]\psi_{\mathrm{h}} \\
\tfrac{1}{2}(1+\gamma_0)\psi_{\mathrm{h}} &= \psi_{\mathrm{h}}\,, \quad \text{“large” components} \\
\mathcal{L}^{(1)} &= \frac{1}{2m_{\mathrm{b}}}\overline{\psi}_{\mathrm{h}}(-\sigma\cdot\mathbf{B} - \tfrac{1}{2}\mathbf{D}^2)\psi_{\mathrm{h}}
\end{aligned}$$

easily derived for smooth (classical) fields (FWT trafo)

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easily derived for smooth (classical) fields (FWT trafo)

(*) equivalent: $\mathcal{L}_{\rm stat} = \overline{\psi}_{\rm h}D_0\psi_{\rm h}$ and $E_{\rm QCD} = E_{\rm stat} + m_{\rm b}$ (universal energy shift)

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- in QCD

$$\Delta m^2 \equiv m_{\mathrm{B}^*}^2 - m_{\mathrm{B}}^2$$

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asymptotics:

$$4\lambda_2^{\mathrm{RGI}} = \lim_{m_\mathrm{b}\to\infty} \left\{ \left[2b_0 \bar{g}^2(m_\mathrm{b}) \right]^{-\gamma_0/2b_0} \Delta m^2 \right\} ,$$

$$\left(\gamma_0 = 3/(8\pi^2) ,\quad b_0 = (11 - \tfrac{2}{3} N_\mathrm{f})/(16\pi^2) \right)$$

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- in HQET

$$\lambda_2^{\mathrm{RGI}} = \tfrac{1}{3}\langle B|\mathcal{O}^{\mathrm{RGI}}_{\mathrm{spin}}|B\rangle \,/\, \langle B|B\rangle$$
$$\mathcal{O}^{\mathrm{RGI}}_{\mathrm{spin}} = \lim_{\mu\to\infty} \left[\, 2b_0\bar{g}^2(\mu)\,\right]^{-\gamma_0/2b_0} \mathcal{O}^{S}_{\mathrm{spin}}(\mu)\,.$$

Here the operator $\mathcal{O}^{S}_{\mathrm{spin}}$ is related to the bare local operator

$$\mathcal{O}_{\mathrm{spin}}(x) = \overline{\psi}_{\mathrm{h}}(x)\frac{1}{2i}F_{kl}(x)\sigma_{kl}\psi_{\mathrm{h}}(x) = \overline{\psi}_{\mathrm{h}}(x)\boldsymbol{\sigma}\cdot\mathbf{B}(x)\psi_{\mathrm{h}}(x)$$

by a multiplicative renormalization depending on the adopted scheme S and a renormalization scale μ – but $\mathcal{O}^{\mathrm{RGI}}_{\mathrm{spin}}$ neither depends on a scheme nor on a scale.

Sin splitting

- at finite mass

$$\Delta m^2 = 2\,\frac{m_{\mathrm{B}^*} + m_{\mathrm{B}}}{M_{\mathrm{b}}}\,C_{\mathrm{spin}}(M_{\mathrm{b}}/\Lambda_{\overline{\mathrm{MS}}})\lambda_2^{\mathrm{RGI}} + \mathrm{O}(1/m_{\mathrm{b}})\,,$$

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from experiment.

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  from experiment.
- $\longrightarrow$ maybe 5-10% precision for B-splitting

static action

- lattice action

$$S_{\rm h}^{\rm W} = a^4 \frac{1}{1 + a\,\delta m_{\rm W}} \sum_x \overline{\psi}_{\rm h}(x)(D_0^{\rm W} + \delta m_{\rm W})\psi_{\rm h}(x)\,.$$

$$P_+\psi_{\rm h} = \psi_{\rm h}\,, \quad \overline{\psi}_{\rm h}P_+ = \overline{\psi}_{\rm h} \quad P_+ = \tfrac{1}{2}(1+\gamma_0)\,.$$

$$D_0^{\rm W}\psi_{\rm h}(x) = \frac{1}{a}\left[\psi_{\rm h}(x) - W^\dagger(x - a\hat{0}, 0)\psi_{\rm h}(x - a\hat{0})\right]\,,$$

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- $W^\dagger(x - a\hat{0}, 0)$: link variable (EH) or
  HYP-links [Hasenfratz & Knechtli; ALPHA Collaboration] (HYP1,HYP2)

Conversion function

- More conventional form of splitting

$$
\begin{aligned}
m_{\mathrm{B}^*}^2 - m_{\mathrm{B}}^2 &= 4\, C_{\mathrm{mag}}^{\mathrm{match}}(m_{\mathrm{b}})\, \lambda_2(m_{\mathrm{b}}) + \mathrm{O}(1/m_{\mathrm{b}}) \\
\lambda_2(m_{\mathrm{b}}) &= \frac{1}{3}\, \langle B|\mathcal{O}_{\mathrm{spin}}^{\overline{\mathrm{MS}}}(\mu = m_{\mathrm{b}})|B\rangle \,/\, \langle B|B\rangle
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  - $\mathcal{O}_{\mathrm{spin}}$ in the $\overline{\mathrm{MS}}$-scheme
  - $m_{\mathrm{b}} \equiv$ pole-mass $m_{\mathrm{Q,b}}$
  - remaining factor $\equiv C_{\mathrm{mag}}^{\mathrm{match}}(m_{\mathrm{b}}) = 1 + C_1 \bar{g}^2(m_{\mathrm{b}}) + \ldots$

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  - $$2m_{\rm Q,b} = m_{\rm B^*} + m_{\rm B} + {\rm O}(\Lambda)$$

Conversion function

- with matrix element of RGI operator

$$\mathcal{O}_{\rm spin}^{\rm RGI} = \lim_{\mu\to\infty} \left[2b_0 \bar{g}^2(\mu) \right]^{-\gamma_0/2b_0} \mathcal{O}_{\rm spin}(\mu) \,.$$

$$\mathcal{O}_{\rm spin}^{\rm RGI} = Z_{\rm spin}^{\rm RGI}(g_0) \mathcal{O}_{\rm spin}$$

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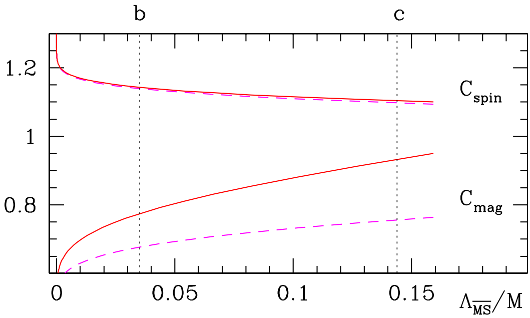
-

$$m_{\rm B^*}^2 - m_{\rm B}^2 = 4C_{\rm mag}(M_{\rm b}/\Lambda_{\overline{\rm MS}})\lambda_2^{\rm RGI} + {\rm O}(1/m_{\rm b})$$

Conversion function

- 

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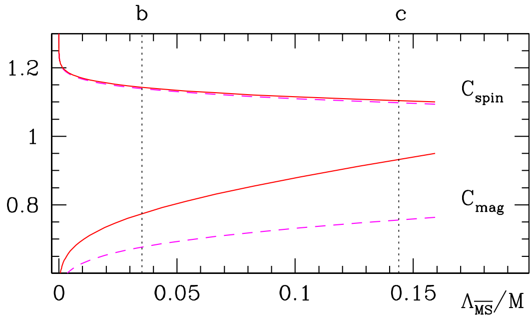
- 

- better "convergence" by changing to

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$$\Delta m^2 = 2\frac{m_{\rm B^*}+m_{\rm B}}{M_{\rm b}} C_{\rm spin}(M_{\rm b}/\Lambda_{\overline{\rm MS}})\lambda_2^{\rm RGI} + {\rm O}(1/m_{\rm b})$$

(pole mass eliminated; renormalon!)

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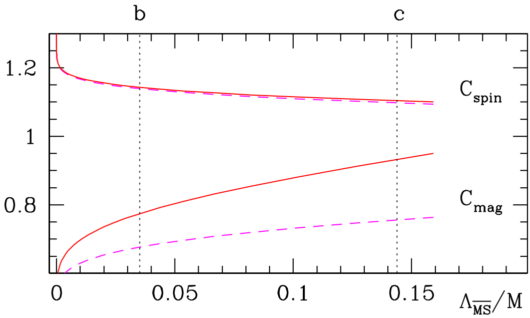
- 

one more order soon: SFB/TR-9

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Needed

- Renormalization factor $Z_{\rm spin}^{\rm RGI}$

  previously 1-loop PT

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  previously 1-loop PT
- bare matrix element (take from literature for now)

Strategy to get $Z^{\mathrm{RGI}}_{\mathrm{spin}}$ $(\mathcal{O}^{\mathrm{RGI}}_{\mathrm{spin}})$

finite volume renormalization scheme $\mathcal{O}^{\mathrm{SF}}_{\mathrm{spin}}$
defined at zero quark mass

$$L_{\max} = \mathrm{const.}/F_\pi = \mathrm{O}(\tfrac{1}{2}\,\mathrm{fm}) : \quad \longrightarrow \quad \mathcal{O}^{\mathrm{SF}}_{\mathrm{spin}}(\mu = 1/L_{\max})$$

$$\downarrow$$

$$\mathcal{O}^{\mathrm{SF}}_{\mathrm{spin}}(\mu = 2/L_{\max})$$

$$\downarrow$$

$$\bullet$$

always $a/L \ll 1$ $\quad\bullet$

$$\bullet$$

$$\downarrow$$

$$\mathcal{O}^{\mathrm{SF}}_{\mathrm{spin}}(\mu = 2^n/L_{\max} = 1/L_{\min})$$

PT: $\downarrow$

$$\mathcal{O}^{\mathrm{RGI}}_{\mathrm{spin}}$$

Result is $\boxed{\mathcal{O}^{\mathrm{RGI}}_{\mathrm{spin}}/\mathcal{O}_{\mathrm{spin}}(L_{\max})}$

combine with $\mathcal{O}_{\mathrm{spin}}(L_{\max}, g_0)$

The SF scheme

- The spin operator

The SF scheme

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  - spin trafo

$$\delta\psi_\mathrm{h} = \sigma_k \psi_\mathrm{h} \,, \qquad \delta\overline{\psi}_\mathrm{h} = -\overline{\psi}_\mathrm{h} \sigma_k \,, \qquad \sigma_k \equiv -\tfrac{1}{2}\epsilon_{ijk}\sigma_{ij}$$

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  - invariance $\longrightarrow$ (local) Noether charge

$$S_k(x) = \frac{1}{1 + a\,\delta m_{\rm W}} \overline{\psi}_{\rm h}(x)\sigma_k W^\dagger(x - a\hat{0}, 0)\psi_{\rm h}(x - a\hat{0})$$

does not renormalize

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- renormalization condition

$$Z_{\rm spin}^{\rm SF}(L)\frac{L^2\langle\, S_1(x+\frac{L}{2}\hat{0})\,\mathcal{O}_{\rm spin}(x)\,\rangle}{\langle\, S_1(x+\frac{L}{2}\hat{0})\,S_1(x)\,\rangle} = \frac{L^2\langle\, S_1(x+\frac{L}{2}\hat{0})\,\mathcal{O}_{\rm spin}(x)\,\rangle}{\langle\, S_1(x+\frac{L}{2}\hat{0})\,S_1(x)\,\rangle}\Bigg|_{g_0=0}\,,$$

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- important detail: boundary conditions such that $F_{kl}\neq 0$ Dirichlet conditions in the 3-direction,

$$U(x,\mu)|_{x_3=0} = \exp(aC)\,,\quad U(x,\mu)|_{x_3=L_3} = \exp(aC')\,,\quad \mu=0,1,2\,,$$

keeping periodic boundary conditions with respect to x_0, x_1, x_2

Abelian fields of “point A” [Lüscher, S., Weisz, Wolff]

$$\begin{aligned} C &= \frac{i}{L}{\rm diag}(\Phi_1,\Phi_2,\Phi_3) = \frac{i}{L}{\rm diag}(-\pi/3,0,\pi/3)\,,\\ C' &= \frac{i}{L}{\rm diag}(\Phi_1',\Phi_2',\Phi_3') = \frac{i}{L}{\rm diag}(-\pi,\pi/3,2\pi/3) \end{aligned}$$

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- integrate out static fields

$$\begin{aligned}\frac{\langle\, S_1(x+\frac{L}{2}\hat{0})\,\mathcal{O}_{\rm spin}(x)\,\rangle}{\langle\, S_1(x+\frac{L}{2}\hat{0})\,S_1(x)\,\rangle} &= \frac{\langle\,\mathrm{Tr}(\mathcal{P}_0(x)B_1(x))\rangle}{\langle\,\mathrm{Tr}(\mathcal{P}_0(x))\,\rangle}, \quad B_1(x)=i\hat{F}_{23}(x),\\ \mathcal{P}_\mu(x) &= W(x,\mu)W(x+a\hat{\mu},\mu)\ldots W(x+(L-a)\hat{\mu},\mu)\end{aligned}$$

remark: spindependent potentials renormalize the same way

- [Eichten & Feinberg; Gromes]: relativistic corrections to static potential

$$V = \ldots + \frac{\mathbf{s}_1 \cdot \mathbf{s}_2}{3 m_1 m_2} V_4(r) + \frac{1}{m_1 m_2} \left[\frac{\mathbf{x} \cdot \mathbf{s}_1 \, \mathbf{x} \cdot \mathbf{s}_2}{r^2} - \frac{\mathbf{s}_1 \cdot \mathbf{s}_2}{3} \right] V_3(r) \, .$$

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- periodic boundary conditions in all directions, $r^2 = x_1^2 + x_2^2 + x_3^2$

$$\frac{x_1 x_2}{r^2} V_3^{\mathrm{SF}}(r,\mu) =$$

$$[Z_{\mathrm{spin}}^{\mathrm{SF}}(1/\mu)]^2 \lim_{L_0 \to \infty} a \sum_{x_0} \frac{\langle \, \mathrm{Tr}(\mathcal{P}_0(0) B_1(0)) \mathrm{Tr}(\mathcal{P}_0(x)^\dagger B_2(x)) \rangle}{\langle \, \mathrm{Tr}\mathcal{P}_0(0) \mathrm{Tr}\mathcal{P}_0(x)^\dagger \, \rangle}$$

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- this should replace the Huntley Michael factor

Renormalization condition with the usual SF

- we may interchange ($x_3 \leftrightarrow x_0$)

$$Z_{\rm spin}^{\rm SF}(L)\frac{L^2\langle\,{\rm Tr}(\mathcal{P}_3(x)E_1(x))\rangle}{\langle\,{\rm Tr}\mathcal{P}_3(x)\,\rangle} = \left.\frac{L^2\langle\,{\rm Tr}(\mathcal{P}_3(x)E_1(x))\rangle}{\langle\,{\rm Tr}\mathcal{P}_3(x)\,\rangle}\right|_{g_0=0}, \text{ at } x_0=\frac{L_0}{2}$$

$$E_1(x) = i\hat{F}_{01}(x)\,, \qquad \text{Dirichlet boundary conditions in time.}$$

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-

$$\frac{L^2\langle\,{\rm Tr}(\mathcal{P}_3(x)E_1(x))\rangle}{\langle\,{\rm Tr}\mathcal{P}_3(x)\,\rangle}\bigg|_{g_0=0}=\frac{\pi}{6}\,\frac{1+\sqrt{3}}{2-\sqrt{3}}+{\rm O}((a/L)^4)\,.$$

Perturbative anomalous dimension

- $$\begin{aligned} \mu \frac{\partial}{\partial \mu} \mathcal{O}_{\rm spin}^{\rm SF} &= \gamma(\bar{g}_{\rm SF}) \mathcal{O}_{\rm spin}^{\rm SF} \,, \\ \gamma^{\rm SF}(\bar{g}_{\rm SF}) &= -\bar{g}_{\rm SF}^2 (\gamma_0 + \gamma_1^{\rm SF} \bar{g}_{\rm SF}^2 + \ldots) \end{aligned}$$

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- $\gamma_1^{\rm SF}$ from
  - $\gamma_1^{\overline{\rm MS}} = (\frac{17}{2} - \frac{13}{12} N_{\rm f})/(32\pi^4)$ [Czarnecki & Grozin; Amoros, Beneke & Neubert]
  - 1-loop relation lat – $\overline{\rm MS}$ [Flynn & Hill]
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- $$\gamma_1^{\rm SF} = \gamma_1^{\overline{\rm MS}} + 2b_0\,\chi^{(1)}_{\rm SF,\overline{MS}} - \gamma_0\chi^{(1)}_{\rm g} = -0.00236 - 0.00352\,N_{\rm f} + 0.00023\,N_{\rm f}^2\,.$$

Non-perturbative running

- step scaling function

$$\mathcal{O}_{\rm spin}^{\rm SF}(\mu) = \sigma_{\rm spin}(\bar{g}^2(1/\mu))\,\mathcal{O}_{\rm spin}^{\rm SF}(2\mu)\,.$$

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$$\begin{aligned} \sigma_{\rm spin}(u) &= \lim_{a/L\to 0} \Sigma_{\rm spin}(u, a/L) \\ \Sigma_{\rm spin}(u, a/L) &= \left.\frac{Z^{\rm SF}_{\rm spin}(2L)}{Z^{\rm SF}_{\rm spin}(L)}\right|_{\bar{g}^2(L)=u\,,\ m=0} \end{aligned}\,.$$

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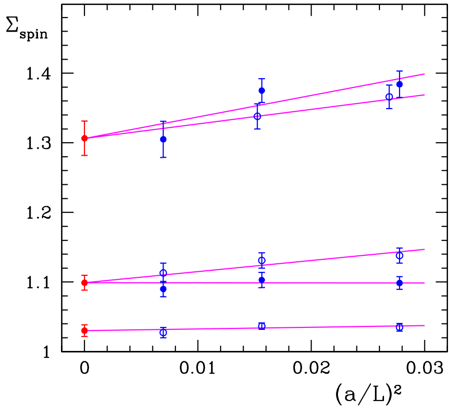
- in perturbation theory (for standard discretization) cutoff effects are small:

$$\delta(u, a/L) = \frac{\Sigma_{\rm spin}(u, a/L) - \sigma_{\rm spin}(u)}{\sigma_{\rm spin}(u)} = \delta_1(a/L)\, u + \mathrm{O}(u^2)\,.$$

$$\delta_1 = \delta_{1,0} + N_{\rm f}\, \delta_{1,1}\,.$$

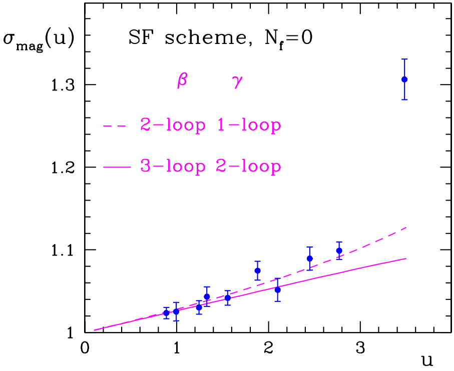
L/a	$\delta_{1,0}(a/L)$	$\delta_{1,1}(a/L)$
6	-0.000236	0.013742
8	-0.000165	0.005791
10	-0.000106	0.003026
12	-0.000072	0.001876
14	-0.000051	0.001296
16	-0.000038	0.000956

Numerical results for $N_f = 0$ (pure gauge)



Filled symbols: $F_{\mu\nu}$ was defined as $\hat{F}_{\mu\nu}$ but with the link variables replaced by HYP2 links

Comparison to perturbation theory

- 

The NP running

- reconstruct through

$$u_0 = \bar{g}^2(L_{\rm max}) = 3.48\,, \quad \sigma(u_{k+1}) = u_k\,, \quad \Rightarrow \quad \bar{g}^2(2^{-k}L_{\rm max}) = u_k\,,$$
$$w_0 = 1\,, \quad w_{k+1} = w_k/\sigma_{\rm spin}(u_{k+1}) \quad \Rightarrow \quad \frac{\Phi_{\rm SF}(2^k/L_{\rm max})}{\Phi_{\rm SF}(1/L_{\rm max})} = w_k\,.$$

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- at $\mu = 2^k/L_{\rm max}\,,\ k \geq 6$

$$\frac{\Phi_{\rm RGI}}{\Phi_{\rm SF}(\mu)} = \left[\,2b_0\bar{g}^2(\mu)\,\right]^{-\gamma_0/2b_0} \exp\left\{-\int_0^{\bar{g}(\mu)} {\rm d}g \left[\frac{\gamma^{\rm SF}(g)}{\beta^{\rm SF}(g)} - \frac{\gamma_0}{b_0 g}\right]\right\}$$

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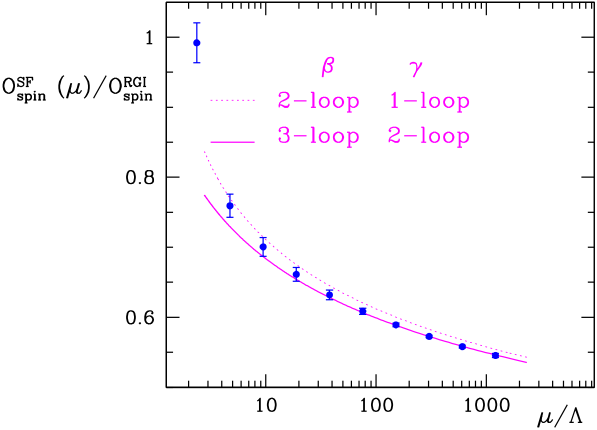
-

$$\Phi_{\rm SF}(\mu)/\Phi_{\rm RGI} \quad = \quad 0.992(29)\,, \quad \text{at } \mu = 1/(2L_{\rm max})\,.$$

The running

▶


$O^{SF}_{spin}(\mu)/O^{RGI}_{spin}$
1
0.8
0.6
10
100
1000
μ/Λ
β
γ
2−loop 1−loop
3−loop 2−loop

The total renormalization factor

- $$\begin{aligned} Z_{\rm spin}^{\rm RGI} &= Z_{\rm spin}^{\rm SF}(L/a, g_0) \times \frac{\Phi_{\rm RGI}}{\Phi_{\rm SF}(1/L)} \quad \text{at} \quad L = 2L_{\rm max} \\ &= 2.62(2) \quad \text{at} \quad \beta = 6.0 \quad \text{EH action} \\ &= 2.63(2) \quad \text{at} \quad \beta = 6.0 \quad \text{HYP2 action} \end{aligned}$$

The spin splitting using baer matrix elements from the literature

- $\beta = 6.0$

$$\text{Ref. [Bochicchio et al., 93]:} \qquad a^2\lambda_2^{\rm bare} = 0.0100(19)\,,$$

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They then quote

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- seems closer to experiment

Conclusions and plans

- The result will be useful to determine the quenched splitting.
  We have precise bare matrix elements at $\beta = 6.0$ and $\beta = 6.2$, but not yet beyond $\longrightarrow$ all-to-all

  Hopefully a continuum extrapolation can be done

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- $N_\mathrm{f} > 0$ seems difficult. Alternative method?