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1 Lattice Fourrier Transformation

1.1 Unitarity of the Fourrier Transformation and free field **1 Pt**

Show that a 1-dimensional Fourrier Transformation is unitary, i.e. the matrix U_{nk} defined by

$$g_n = \sum_k U_{nk}\,\tilde{g}_k\,, \tag{1.1}$$

with

$$\begin{aligned} g_n &= f(x = na)\,, &(1.2)\\ \tilde{g}_k &= \sqrt{N}\tilde{f}(p = k\frac{2\pi}{L})\,,\ N = L/a &(1.3)\\ f(x) &= \sum_p \mathrm{e}^{ipx}\tilde{f}(p) &(1.4)\end{aligned}$$

is unitary,

$$U^\dagger U = 1\,. \tag{1.5}$$

Why does this hold also for the d-dimensional Fourrier transformation?
Assuming the transformation in a finite volume, $V = L_0L_1L_2L_3$, show that

$$\frac{1}{V}\sum_p \mathrm{e}^{ipx} = \delta(x) \tag{1.6}$$

with the lattice delta-function, defined by

$$a^4\sum_x \delta(x-y)\,f(x) = f(y)\,. \tag{1.7}$$

Take the infinite volume limit to show

$$\int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}^4p}{(2\pi)^4}\,\mathrm{e}^{ipx} = \delta(x)\,. \tag{1.8}$$

1.2 Free scalar propagator **2.5 Pt**

The matrix M is defined by

$$\begin{aligned} S_{\text{free}} &= a^4\sum_x\{-\phi^*(x)\partial^*_\mu\partial_\mu\phi(x) + m^2|\phi|^2(x)\} &(1.9)\\ &= \phi^\dagger M\phi\,,\ \phi_n = a\phi(na)\,. &(1.10)\end{aligned}$$

Write down M explicitly in one dimension (finite L) and observe that it is hermitian.

Show that M is diagonalized by the Fourrier transformation matrix,

$$U^\dagger M U = \tilde{M}\ ,\ \ \tilde{M}_{kk'} = \delta_{kk'}\mu_k^2 \tag{1.11}$$

and obtain the elements of μ_k^2. Expand μ_k^2/a^2 up to (and including) second order in the lattice spacing a at fixed momenta $p = k\frac{2\pi}{L}$. Write down the explicit form for the Green function, i.e. the inverse of M and its a-expansion.

1.3 Three- and four-point functions **0.5 Pt**

What are the three-point functions

$$G_3(x,y,z) = \langle \phi(x)\phi^*(y)\phi(z)\rangle \tag{1.12}$$
$$G_3'(x,y,z) = \langle \phi(x)\phi^*(y)\phi^*(z)\rangle \tag{1.13}$$

and the four-point function

$$G_4(x,y,z,r) = \langle \phi(x)\phi^*(y)\phi(z)\phi^*(r)\rangle\,? \tag{1.14}$$

2 Symanzik improvement of the scalar theory

Based on an analysis of the structure of lattice perturbation theory, K. Symanzik has argued [1] that the a-effects of a lattice theory of order a^n can be removed by adding to the lattice Lagrangian density a complete set of local fields $O_i(x)$ which

- are invariant under all the symmetry transformations of the original Lagrangian
- have mass dimension $[O_i(x)] \equiv d_O \leq n + d$ in d space-time dimensions
- are *not* equivalent by the equations of motion of the theory.

The improved theory has the action

$$S_{\rm impr} = S + \delta S\,, \quad \delta S = a^d \sum_x \sum_i c_i(\lambda) a^{d_O - d} O_i(x)\,, \tag{2.1}$$

The statement refers to so-called on-shell quantities, in particular energies of the theory and means that at fixed $\mathbf{p}, m_{\rm R}$ terms of order a^n are absent in the expansion in a.
Local fields are composed of $\phi(x), \partial_\mu\phi(x), \partial_\mu\partial_\nu\phi(x), \ldots, m$. The statement holds after renormalization (replacement of bare parameters by observables).

2.1 Automatic O(a) improvement **0.5 Pt**

Why are there no linear a-effects in the ϕ^4 theory with standard action in $d = 4$ dimensions? Is this true in $d = 3$ dimensions?

2.2 O(a^2) improvement **1 Pt**

Make a complete list of all allowed local fields for $n = 2$ (i.e. a^2 improvement) for $d = 4$. Start with a general list of the interacting theory ($\lambda > 0$).

Now turn to the free theory. Use the equation of motion $(-\partial_\mu\partial_\mu + m^2)\phi(x) = 0$ to show that

$$O_1(x) = \sum_{\mu=0}^{3} [\partial_\mu\partial_\mu\phi(x)]^2\,, \tag{2.2}$$

$$O_2(x) = [\sum_{\mu=0}^{3} \partial_\mu\partial_\mu\phi(x)\,]^2 \tag{2.3}$$

is a complete list. Note that as the free theory is quadratic in ϕ, there are no terms with more than two ϕ-fields.

2.3 Energy momentum relation **2.5 Pt**

Compute the energy momentum relation of the free theory as a function of c_1, c_2.

We remark that the improvement of correlation functions requires additional improvement terms of the fields appearing in the correlation functions [2]. This is avoided above by just looking at the energies.

References

[1] K. Symanzik, *Continuum limit and improved action in lattice theories. 1. Principles and ϕ^4 theory*, *Nucl. Phys.* **B226** (1983) 187.

[2] M. Lüscher, S. Sint, R. Sommer, and P. Weisz, *Chiral symmetry and O(a) improvement in lattice QCD*, *Nucl. Phys.* **B478** (1996) 365–400, [`hep-lat/9605038`].

3 The group SU(N)

SU(N) matrices U satsify $U\,U^\dagger = \mathbb{1}\,,\ \det U = 1$.
They can be written as

$$U = \mathrm{e}^B\,,\ B = -B^\dagger\,,\ \operatorname{tr} B = 0\,.$$

3.1 **0.5 Pt**

Show that the metric tensor

$$G_{ab} = \operatorname{tr}\,(\frac{\partial W}{\partial \omega^a}\,\frac{\partial W^\dagger}{\partial \omega^b})$$

is symmetric and positive.

3.2 SU(2) **1.5 Pt**

Show that in SU(2) we can parametrize

$$B = \sum_{a=1}^{3} T^a B^a\,,\ T^a = \frac{\tau^a}{2i}\,,$$

where τ^a are the Pauli matrices.

What are the structure constants f^{abc} of SU(2)?

Show that

$$W = \exp(\sum_{a=1}^{3} T^a B^a)$$

can be written as

$$\begin{aligned} W &= w^0 + i w^k \tau^k\,, \quad w^\mu \in \mathbb{R}\,,\ w^\mu w^\mu = 1\,, \\ W &= \sqrt{1-\omega^a\omega^a} + i\omega^a\tau^a\,, \quad \omega^a \in [0,1]\,. \end{aligned}$$

Compute the metric tensor in the latter parametrization and write down the integration measure in the former one.

3.3 Group integrals **1.5 Pt**

We consider the group integral

$$f_{ijkl} = \int \mathrm{d}U\, U_{ij} (U^\dagger)_{kl}$$

and define

$$\text{fixed } j,k: \quad F_{il} = f_{ijkl}\,.$$

Invariance of the Haar measure gives

$$\Lambda F \Lambda^{-1} = F \quad \forall\,\Lambda \in \mathrm{SU}(N)\ .$$

Show that any complex $N \times N$ matrix with this property is a multiple of the unit matrix, $F = c \times \mathbb{1}$.
Hint: first consider SU(2) and then imbed SU(2) in SU(N).

3.4 Plaquette gauge action **0.5 Pt**

Show that the plaquette gauge action is real.

4 Transfer matrix

4.1 Static scalar quarks

The static action for scalar quarks is

$$S_\phi^{\rm stat}[\phi, U] \;=\; a^4 \sum_x \phi^\dagger(x)[-D_0^* \, D_0 + m_0^2]\phi(x)\,. \tag{4.1}$$

What is the transfer matrix of this theory without gauge fields ($U(x,\mu) = \mathbb{1}\;\forall\; x, \mu$)?

What is the transfer matrix of this theory including gauge fields described by the (Wilson) plaquette gauge action.

4.2 Pure gauge theory

We consider the transfer matrix in temporal gauge ($U(x,0) = \mathbb{1}\;\forall\; x$),

$$\hat{\mathbb{T}}\,\psi[V] \;=\; \int \mathrm{D}V'\, K[V,V']\,\psi[V'] \tag{4.2}$$

$$K[V,V'] \;=\; \mathrm{e}^{-\Delta S[V,\mathbb{1},V']}\,, \tag{4.3}$$

$$\Delta S[V,W,V'] \;=\; \tfrac{1}{2}S_p[V] + S_k[V,W,V'] + \tfrac{1}{2}S_p[V']\,, \tag{4.4}$$

$$S_p[V] \;=\; \frac{1}{g_0^2}\sum_{\mathbf{x}}\sum_{k,l=1}^{3} \;\mathrm{tr}\, P(x,k,l)\,. \tag{4.5}$$

$$S_k[V,W,V'] \;=\; \frac{1}{g_0^2}\sum_{\mathbf{x}}\sum_{k=1}^{3} \;\mathrm{tr}\left[P(x,k,0) + P(x,k,0)^\dagger\right]\,, \tag{4.6}$$

$$P(\mathbf{x},k,0) = 1 - V'(\mathbf{x},k)\,W(\mathbf{x}+a\hat{k})\,V(\mathbf{x},k)^{-1}\,W(\mathbf{x})^{-1}\,. \tag{4.7}$$

Show that it is a positive semidefinite operator:

$$\langle\psi|\hat{\mathbb{T}}|\psi\rangle = \int \mathrm{D}[V](\psi[V])^*\,\hat{\mathbb{T}}\psi[V] \geq 0 \;\forall\; \psi \text{ with } \langle\psi|\psi\rangle > 0\,.$$

Hints:

- Separate the potential term S_p and formulate the problem in terms of the kinetic part only.
- Factorize the kinetic part into single link transfer matrices.
- Perform a (convergent) expansion in $\frac{1}{g_0^2}$ of the single link transfer matrix to show it is positive semidefinite.

Remark: with some more mathematics one can also show that the single link transfer matrix is strictly positive.

4.3 Static scalar fields II

Now perform the same discussion for the transfer matrix of static scalar fields to show its positivity.

5 Dirac Fermions

5.1 Free continuum fermions

Evaluate the free fermion timeslice correlator,

$$S(x_0;\mathbf{p}) \;=\; \int \frac{\mathrm{d}p_0}{(2\pi)} \mathrm{e}^{ip_0x_0}\,[ip_\mu\gamma_\mu + m]^{-1}\,. \tag{5.1}$$

Discuss in particular the case $\mathbf{p}=0$. What does it tell us about the number of particles which are there?

5.2 Free Wilson fermions

The dispersion relation is given by the solution of

$$2\cosh(aE) = \frac{1+a^2\tilde{\mathbf{p}}^2}{A} + A\,, \quad A = 1 + am + \tfrac{1}{2}a^2\hat{\mathbf{p}}^2\,. \tag{5.2}$$

Show that after a proper renormalization

$$m \to E(\mathbf{p}=0) \equiv m_{\mathrm{pol}} \tag{5.3}$$

we have

$$E(m_{\mathrm{pol}},\mathbf{p}) = \sqrt{m_{\mathrm{pol}}^2 + \mathbf{p}^2} + \mathrm{O}(a^2)\,. \tag{5.4}$$

Free Wilson fermions have discretization errors $\mathrm{O}(a^2)$. Note that for QCD this is not true in general, but only after adding one improvement term to the action.

5.3 Symmetries of Wilson fermions

Show that parity, charge conjugation, time reversal are symmetries of the Wilson fermion action. Show also that the Wilson Dirac operator is γ_5-hermitian,

$$\gamma_5(D_W+m)\gamma_5 \;=\; (D_W+m)^\dagger\,. \tag{5.5}$$

Which of these symmetries is violated by a twisted mass term?

Show further that the theory with a twisted mass term has a (“spurionic”) symmetry where together with the fields a parameter in the action is changed:

$$\psi \;\to\; \tau^1\psi\,, \quad \overline{\psi} \to \overline{\psi}\tau^1 \tag{5.6}$$

$$m \;\to\; -m\,. \tag{5.7}$$

In the literature, usually μ is used instead of m to denote the twisted mass.