Introduction to lattice gauge theories

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We give an introduction to lattice gauge theories with an emphasis on QCD. Requirements are quantum mechanics and for a better understanding relativistic quantum mechanics and continuum quantum field theory.

These are not lecturenotes written to be easily readable (a script), but my private notes.

Still I am of course happy to receive corrections.

References

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Contents

1 Introduction

1.1 Particle physics

It is about the fundamental forms of matter (quarks = constituents of nucleons and their relatives) and their interactions. About scattering processes (LHC!) but also about the bound states: HOW is a nucleon made up of quarks.

The fundamental formulation is a quantum field theory (or string theory, which for energies far below M_{Planck} is again a quantum field theory). Field theories combine Poincare invariance and quantum mechanics.

1.2 Why a lattice?

Field theories have coupling "constants", e.g. the fine structure constant α in quantum electro dynamics (QED). The standard continuum treatment is an expansion in these coupling constants: "perturbation theory".

QCD is the part of the theory which describes the by far dominant interactions of quarks (up,down,charm,strange,top,bottom). It has a rather large effective coupling constant at distances of the order (0.1 fm to 1 fm).

- There are phenomena, in particular in QCD, which are intrinsically non-perturbative:

  confinement: quarks are never observed as free particles

$$m_{\mathrm{proton}} = m_{\mathrm{p}} \gg 2m_{\mathrm{u}} + m_{\mathrm{d}} \tag{1.1}$$

  In fact in the limit of vanishing quark masses:

$$m_{\mathrm{p}} \approx m_{\mathrm{p}}|_{m_{\mathrm{u}}=m_{\mathrm{d}}=0} \sim \mu\,\mathrm{e}^{-\mathrm{const.}/\alpha_{\mathrm{s}}(\mu)} = 0 \quad \text{to all orders in } \alpha_{\mathrm{s}}: \quad 0 + 0\alpha + 0\alpha^2 + \ldots \tag{1.2}$$

- the hadron mass spectrum is non-perturbative and can in principle be computed depending on just a few parameters (neglecting electromagnetism, weak interaction, gravitation):

$$\alpha_{\mathrm{s}},\ m_{\mathrm{u}},\ m_{\mathrm{d}},\ m_{\mathrm{s}},\ m_{\mathrm{c}},\ m_{\mathrm{b}},\ m_{\mathrm{t}}$$

- The lattice formulation is designed to enable such computations

- The lattice formulation is the only known formulation which contains perturbative and non-perturbative "sectors".
  It is therefore to be regarded as THE formulation (definition) of a quantum field theory, in particular QCD.

1.3 Where will we go?

- Introduce the ingredients
  - path integral, Euclidean time
  - lattice (scalar fields)
  - gauge fields (gluons)
  - fermions (quarks)

- Discuss concepts and computational methods
  - continuum limit, Symanzik effective theory
  - strong coupling expansion
  - MC method
  - multiscale methods (maybe in part II)
- Discuss some results, e.g.

  quark — anti-quark potential

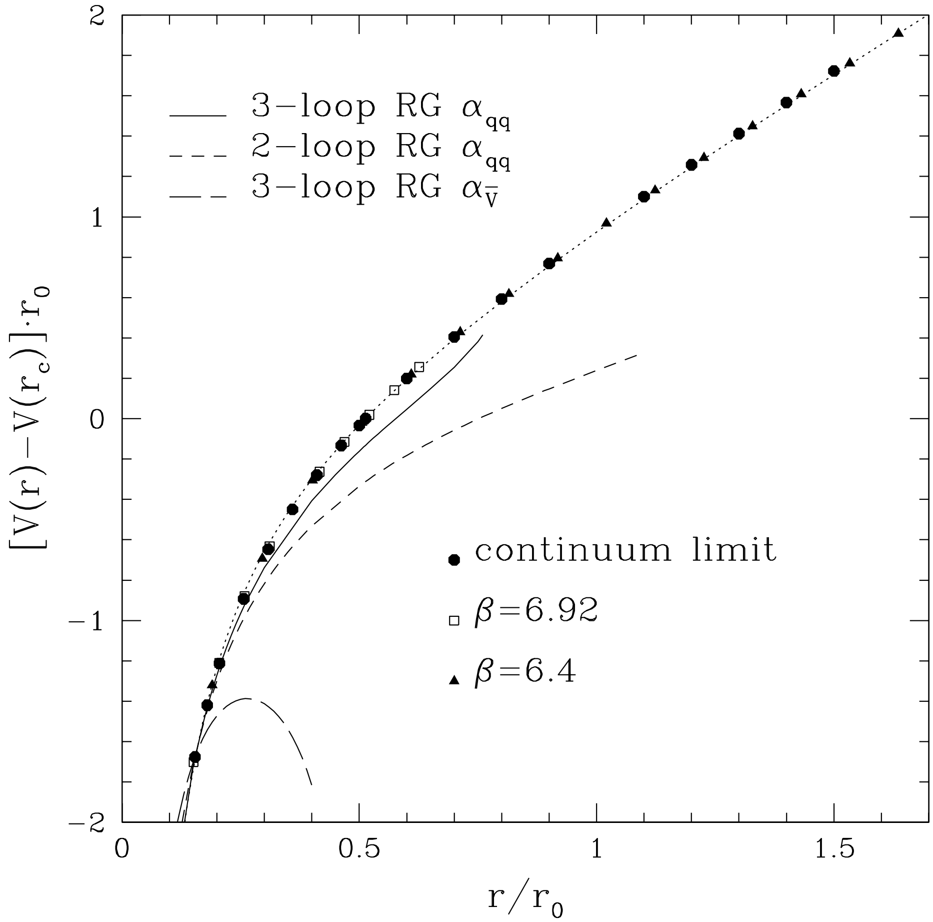


Figure 1: *Static quark potential in the pure gauge theory. [2001]*

glueball spectrum pure gauge theory

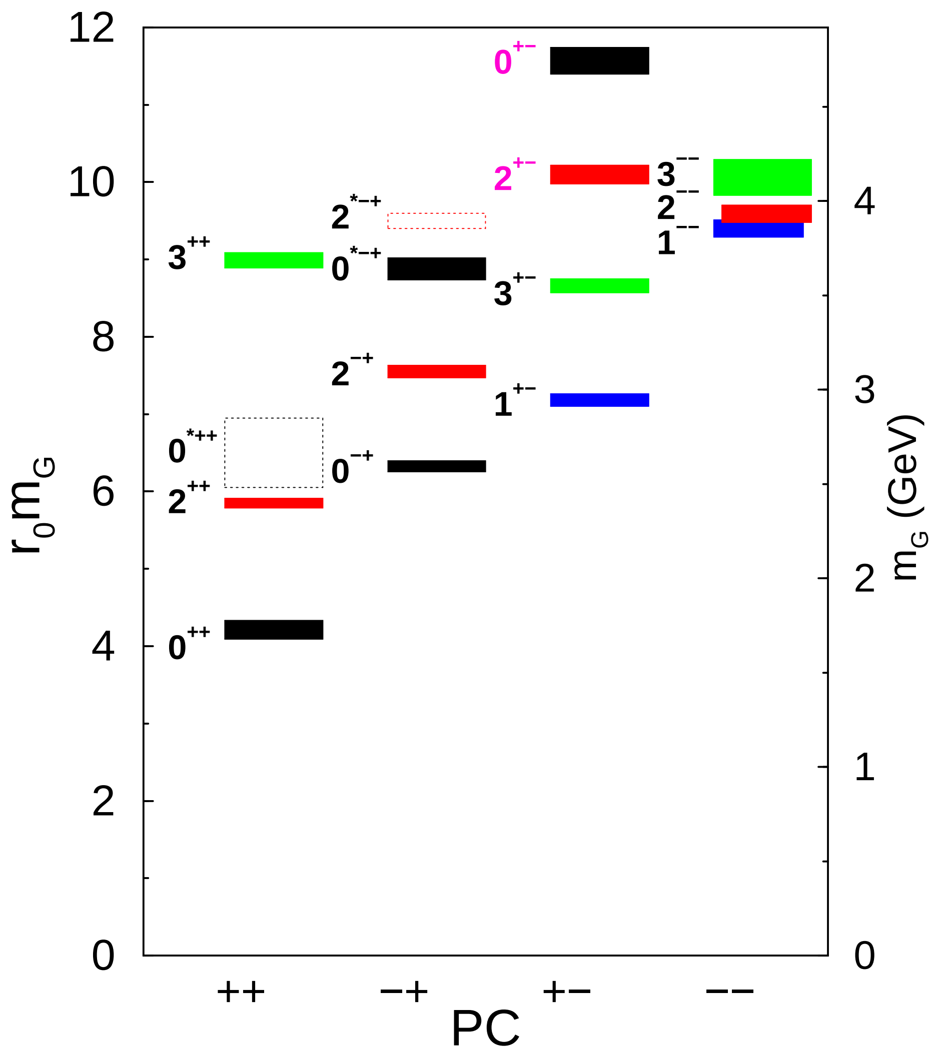


Figure 2: *Glueball spectrum in the pure gauge theory. [1998]*

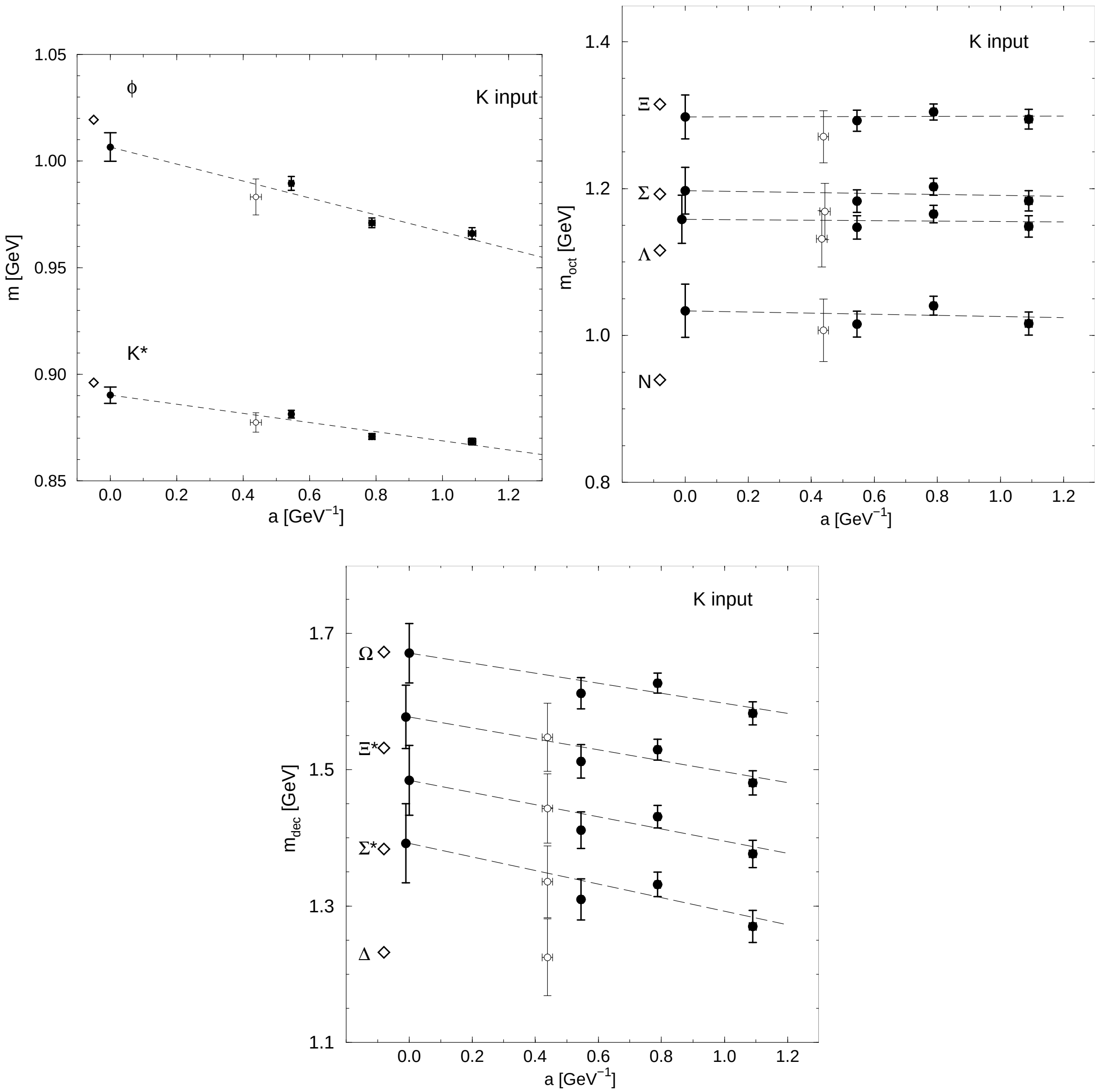


Figure 3: *Hadron mass spectrum. [2001]*

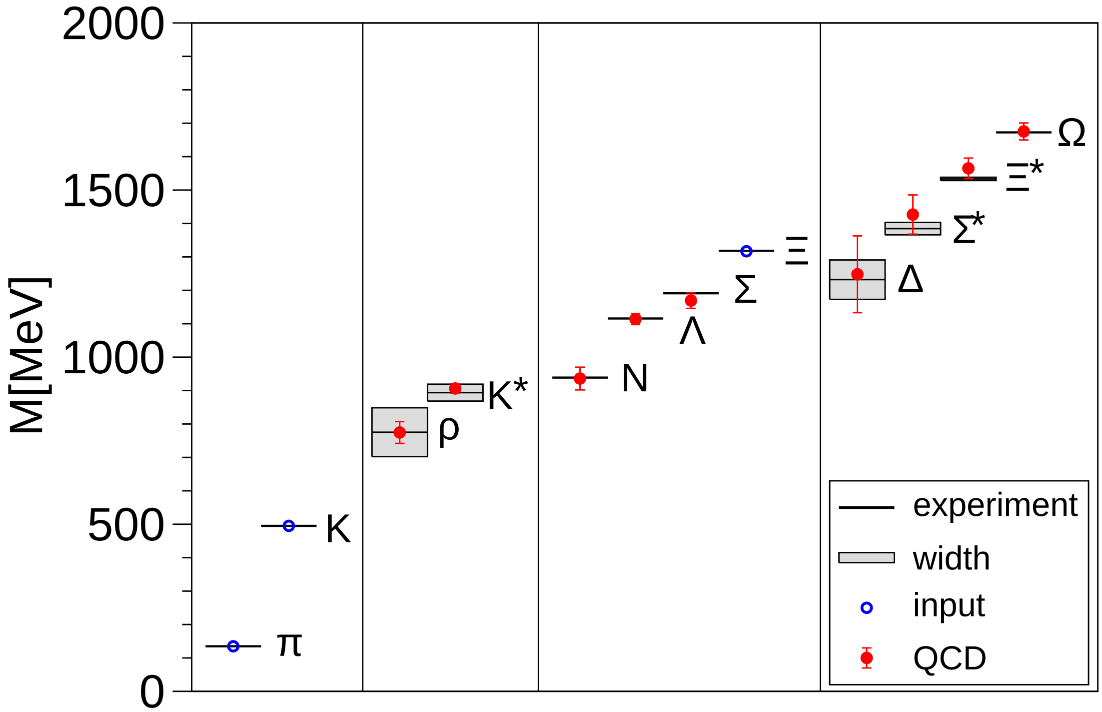


Figure 4: *Hadron mass spectrum. [2009]*

chiral symmetry breaking

decay rates: $\pi \to \mu\nu$

[running of QCD coupling (maybe in part II)]

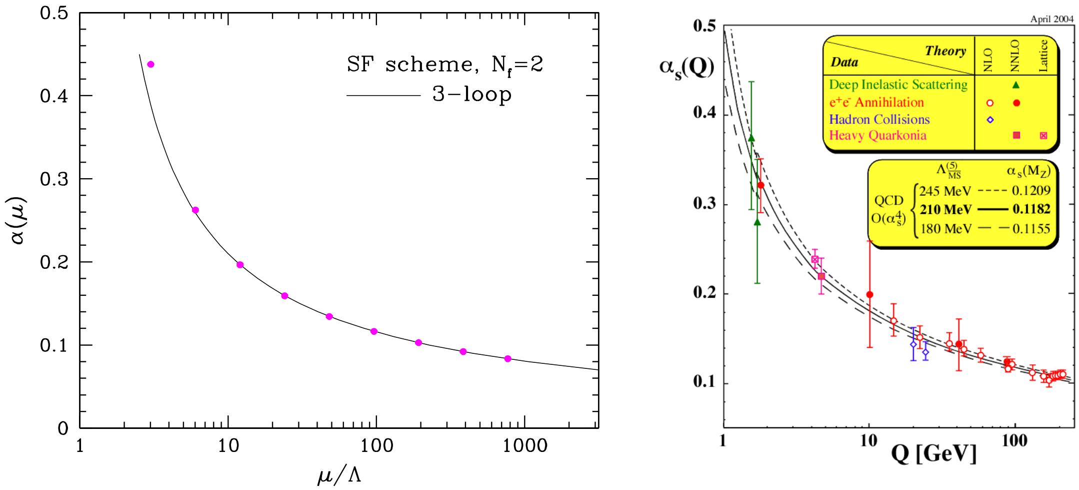


Figure 5: *α_s: left: lattice gauge theory [2005], right: experiment + PT [2004].*

[interplay with effective theories (maybe in part II)
i.e. e.g. expansion in a or $1/m_{\rm b}$]

[elastic scattering phases (maybe in part II)]

2 Pathintegral in quantum mechanics

2.1 Euclidean Green functions in QM

We start from a QM Hamiltonian of one degree of freedom

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \qquad h/(2\pi) = 1, c = 1\,. \tag{2.1}$$

The statistical mechanics quantum partition function can be written as (T=1/temperature!)

$$Z = \operatorname{Tr} \mathrm{e}^{-T\hat{H}} = \int_{q(0)=q(T)} \mathrm{D}[q]\, \mathrm{e}^{-S[q]} \tag{2.2}$$

(we will "show" that). The Euclidean action

$$S[q] = \int_0^T \mathrm{d}t\, \underbrace{(\frac{m}{2}\,\dot{q}^{\,2} + V(q))}_{=\mathcal{L}|_{t\to it}}, \quad \mathrm{D}[q] = \text{“}\prod_t \mathrm{d}q(t)\text{“} \tag{2.3}$$

A "derivation":

We define the Euclidean time evolution operator (= transfer matrix) by a small (infinitesimal) time unit a,

$$\hat{\mathbb{T}} = \mathrm{e}^{-a\hat{H}}\,, \quad \mathbb{T}(q,q') = \langle q|\mathrm{e}^{-a\hat{H}}|q'\rangle\,, \tag{2.4}$$

such that

$$Z = \operatorname{Tr}\hat{\mathbb{T}}^N\,, \quad N = T/a \tag{2.5}$$

$$Z = \int \mathrm{d}q_0\, \mathrm{d}q_1 \ldots \mathrm{d}q_{N-1}\, \mathbb{T}(q_0,q_1)\, \mathbb{T}(q_1,q_2)\, \ldots\, \mathbb{T}(q_{N-1},q_0) \tag{2.6}$$

Use the BCH formula (or explicitly) to show

$$\mathbb{T}(q,q') = \langle q|\mathrm{e}^{-aV(\hat{q})-a\frac{1}{2m}\hat{p}^2}|q'\rangle \tag{2.7}$$

$$= \langle q|\mathrm{e}^{-aV(\hat{q})/2}\mathrm{e}^{-a\frac{1}{2m}\hat{p}^2}\mathrm{e}^{-aV(\hat{q})/2}|q'\rangle + \mathrm{O}(a^3) \tag{2.8}$$

$$[\quad \langle q|\mathrm{e}^{-aV(\hat{q})/2}|q'\rangle = \mathrm{e}^{-a\frac{1}{2}V(q)}\delta(q-q') \tag{2.9}$$

$$\langle p|p'\rangle = \delta(p-p')\,, \tag{2.10}$$

$$\langle q|p\rangle = \frac{1}{(2\pi)^{1/2}}\mathrm{e}^{-ipq} \tag{2.11}$$

$$\langle q|\mathrm{e}^{-a\frac{1}{2m}\hat{p}^2}|q'\rangle = \int \mathrm{d}p\, \langle q|p\rangle\langle p|q'\rangle \mathrm{e}^{-a\frac{1}{2m}p^2} \tag{2.12}$$

$$= \frac{1}{2\pi}\int \mathrm{d}p\mathrm{e}^{-ip(q-q')}\mathrm{e}^{-a\frac{1}{2m}p^2} = \frac{1}{2\pi}\sqrt{\frac{2\pi m}{a}}\mathrm{e}^{-m\frac{1}{2a}(q-q')^2} \tag{2.13}$$

$$] \tag{2.14}$$

$$= \sqrt{\frac{m}{2\pi a}}\mathrm{e}^{-a[\frac{1}{2}(V(q)+V(q'))+\frac{m}{2}(\frac{q-q'}{a})^2]} + \mathrm{O}(a^3) \tag{2.15}$$

And with

$$\frac{q(t+a)-q(t)}{a} = \dot{q}\,(t+a/2) + \mathrm{O}(a^2) \tag{2.16}$$

we have naively shown the equivalence of quantum mechanics and a 1-dimensional Euclidean pathintegral. [remark about unboundedness, therefore a-expansion is not obvious]

$$Z = \int \mathrm{d}q_0\,\mathrm{d}q_1 \ldots \mathrm{d}q_{N-1}\,\mathbb{T}(q_0,q_1)\,\mathbb{T}(q_0,q_1)\,\ldots\,\mathbb{T}(q_{N-1},q_0) \tag{2.17}$$

$$[\prod_t \mathrm{d}q(t) = \prod_i q(t_i) = \mathrm{D}[q] \tag{2.18}$$

$$] \tag{2.19}$$

$$= const. \times \int \mathrm{D}[q] \exp\left(-\frac{a}{2}\sum_{i=0}^{N-1}(\frac{q_{i+1}-q_i}{a})^2 + V(q_i)\right), \quad q_N = q_0 \tag{2.20}$$

$$= const. \times \int \mathrm{D}[q] \exp(-S[q]) \quad \text{approximated on a 1-d lattice} \tag{2.21}$$

$$\xrightarrow{a\to 0} const. \times \int \mathrm{D}[q] \exp(-S_{\text{cont}}[q])\,. \tag{2.22}$$

Euclidean Green functions:

$$t_1 \geq t_2 \tag{2.23}$$

$$G(t_1,t_2) = \frac{1}{Z}\int \mathrm{D}[q]\,\mathrm{e}^{-S[q]}\,q(t_1)\,q(t_2)\,, \quad n_i = t_i/a \tag{2.24}$$

$$= \frac{1}{Z}\int \mathrm{d}q_0\,\mathrm{d}q_1 \ldots \mathrm{d}q_{N-1}\,\mathbb{T}(q_0,q_1)\,\mathbb{T}(q_1,q_2)\,\ldots\,\mathbb{T}(q_{N-1},q_0)\,q_{n_1}\,q_{n_2} \tag{2.25}$$

$$= (\,\mathrm{Tr}\,\hat{\mathbb{T}}^{T/a})^{-1}\ \mathrm{Tr}\,\hat{\mathbb{T}}^{(T-t_1)/a}\,\hat{q}\,\hat{\mathbb{T}}^{(t_1-t_2)/a}\,\hat{q}\,\hat{\mathbb{T}}^{t_2/a} \tag{2.26}$$

$$= \frac{\mathrm{Tr}\ \sum_n |n\rangle\langle n|\mathrm{e}^{-E_n(T-t_1)}\,\hat{q}\,\mathrm{e}^{-\hat{H}\,(t_1-t_2)}\,\hat{q}\,\mathrm{e}^{-\hat{H}\,t_2}}{\mathrm{Tr}\ \sum_n |n\rangle\langle n|\mathrm{e}^{-E_n\,T}} \tag{2.27}$$

$$= \frac{\sum_n \langle n|\mathrm{e}^{-E_n(T-t_1)}\,\hat{q}\,\mathrm{e}^{-\hat{H}\,(t_1-t_2)}\,\hat{q}\,\mathrm{e}^{-\hat{H}\,t_2}|n\rangle}{\sum_n \mathrm{e}^{-E_n\,T}} \tag{2.28}$$

$$[\text{ the ground state is non-degenerate }] \tag{2.29}$$

$$\xrightarrow{T\to\infty} \langle 0|\mathrm{e}^{\hat{H}\,t_1}\hat{q}\mathrm{e}^{-\hat{H}\,t_1}\mathrm{e}^{\hat{H}\,t_2}\hat{q}\mathrm{e}^{-\hat{H}\,t_2}|0\rangle \tag{2.30}$$

$$= \langle 0|\hat{q}(t_1)\,\hat{q}(t_2)|0\rangle \tag{2.31}$$

$$\hat{q}(t) = \mathrm{e}^{\hat{H}\,t}\hat{q}\mathrm{e}^{-\hat{H}\,t} = \text{Euclidean Heisenberg operators} \tag{2.32}$$

And more generally:

$$t_1 \geq t_2 \geq t_3 \ldots, \quad T \to \infty \tag{2.33}$$

$$G(t_1,t_2,...,t_n) = \frac{1}{Z}\int \mathrm{D}[q]\,\mathrm{e}^{-S[q]}q(t_1)\,q(t_2)\,\ldots\,q(t_n)\,. \tag{2.34}$$

$$= \langle 0|\hat{q}(t_1)\,\hat{q}(t_2)\,\ldots\hat{q}(t_n)|0\rangle \tag{2.35}$$

$$\text{general}\ \ t_i\,, \qquad T \to \infty \tag{2.36}$$

$$G(t_1,t_2,...,t_n) = \langle 0|\mathcal{T}\,\{\hat{q}(t_1)\,\hat{q}(t_2)\,\ldots\hat{q}(t_n)\}\,|0\rangle \tag{2.37}$$

In lattice gauges theories, Euclidean Green functions = correlation functions = correlators are the central objects (but for 1+3=4 dimensions). They are mostly computed numerically by a Monte Carlo process. Assume one has the Euclidean Green functions. Real time physics is obtained (in principle) by analytic continuation (see later). But also directly:

$(T \to \infty)$

$$
\begin{aligned}
G(t_1, t_2) &= G(t_1 - t_2) = \langle 0|\mathrm{e}^{\hat{H}\,t_1}\hat{q}\mathrm{e}^{-\hat{H}\,t_1}\mathrm{e}^{\hat{H}\,t_2}\hat{q}\mathrm{e}^{-\hat{H}\,t_2}|0\rangle && (2.38)\\
&= \langle 0|\hat{q}\mathrm{e}^{-\hat{H}\,(t_1-t_2)}\hat{q}|0\rangle \mathrm{e}^{E_0\,(t_1-t_2)} && (2.39)\\
&= \langle 0|\hat{q}\sum_n |n\rangle \mathrm{e}^{-E_n\,(t_1-t_2)}\langle n|\,\hat{q}|0\rangle\,\mathrm{e}^{E_0\,(t_1-t_2)} && (2.40)\\
&= \sum_n \alpha_n^2 \mathrm{e}^{-(E_n-E_0)\,(t_1-t_2)}\,, \quad \alpha_n = \langle n|\hat{q}|0\rangle && (2.41)
\end{aligned}
$$

from the large $t_2 - t_1$ behaviour: $E_1, E_2, \ldots$. Improvement of precision by different $f(\hat{q})$.

In various places we dropped $\mathrm{O}(a^2)$ terms. The result of a lattice path integral is only unique (universal) in the limit $a \to 0$.

2.2 Quantum field theory

Let us give a very rough overview what QFT (and therefore QCD) is about.

It describes

- particles moving in space and time $|\mathbf{p}\rangle$
- multiparticle states $|\mathbf{p}_1, \mathbf{p}_2\rangle$:

$$\hat{H}|\mathbf{p}_1, \mathbf{p}_2\rangle \approx (\sqrt{\mathbf{p}_1^2 + m_1^2} + \sqrt{\mathbf{p}_2^2 + m_2^2})|\mathbf{p}_1, \mathbf{p}_2\rangle \quad (2.42)$$

  $\approx$ because there is always some (small) interaction

- scattering, decays, particle creation
- bound states $\to$ one of the main subjects of LGT.

Construction of QFT (scalar particles):
Creation and annihilation of particles at any point in space and time: QM operators at any point, quantum fields:

$$
\begin{aligned}
\hat{q}_i &\to \hat{\phi}(\mathbf{x}) && (2.43)\\
\hat{p}_i &\to \hat{\pi}(\mathbf{x}) && (2.44)\\
& [\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = 0 = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] && (2.45)\\
& [\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) && (2.46)
\end{aligned}
$$

$\mathbf{x}$ not the QM variable, but more an index, labelling dof's. Free particle:

$$\hat{H} = \int \mathrm{d}^3\mathbf{x}\{\tfrac{1}{2}\hat{\pi}(\mathbf{x})^2 + \tfrac{1}{2}\partial_j\hat{\phi}(\mathbf{x})\partial_j\hat{\phi}(\mathbf{x}) + \frac{m^2}{2}\hat{\phi}^2(\mathbf{x})\} \quad (2.47)$$

Interactions: very restricted by general principles:

- unitarity, causality

- renormalizability → locality, dimension of fields in the Lagrangian

Only

$$\mathcal{L}_{\text{int}}(x) = \frac{\lambda_0}{4}\phi^4(x) \quad \rightarrow \quad \hat{H}_{\text{int}} = \frac{\lambda_0}{4}\hat{\phi}^4(\mathbf{x}) \tag{2.48}$$

is possible. There is only one parameter: λ_0, dimensionless.

Let us take two fields, combine them into one complex field, corresponds in the end to a charged particle and its anti-particle.

$$\phi = \frac{1}{\sqrt{2}}[\phi_1 + i\phi_2]\,, \quad |\phi|^2 = \tfrac{1}{2}(\phi_1^2 + \phi_2^2)\,, \quad \mathcal{L}_{\text{int}}(x) = \frac{\lambda_0}{4}|\phi(x)|^4 \tag{2.49}$$

The formal continuum path integral:

$$Z = \int \mathrm{D}[\phi]\mathrm{D}[\phi^\dagger]\,\mathrm{e}^{-S[\phi]}\,, \tag{2.50}$$

$$S[\phi] = \int \mathrm{d}^4(x)\,\{\partial_\mu\phi^*(x)\partial_\mu\phi(x) + m^2|\phi(x)|^2 + \frac{\lambda_0}{4}|\phi(x)|^4\} \tag{2.51}$$

$$G(x,y) \equiv \langle\phi(x)\phi^*(y) \equiv Z^{-1}\int \mathrm{D}[\phi]\mathrm{D}[\phi^*]\,\mathrm{e}^{-S[\phi]}\phi(x)\phi^*(y)\,. \tag{2.52}$$

2.2.1 Problems with the path integral

It is a rather formal object:

- $\lambda_0 = 0$: gaussian integrals as in QM
- $\lambda_0 > 0$: perturbation theory (PT) in λ,

$$G(x,y): \tag{2.53}$$

$$G = G^{(0)} + \lambda_0 G^{(1)} + \lambda_0^2 G^{(2)} + \ldots \tag{2.54}$$

  again gaussian integrals ⟶Feynman diagrams

- $\lambda_0 < 0$: singular:

$$\int_{-\infty}^{\infty} \mathrm{e}^{+|\lambda_0|\phi^4(x)} \rightarrow \infty \tag{2.55}$$

  PT has a zero radius of convergence: asymptotic series

- Divergences: start with m_0, λ_0, compute

$$E^2 = \mathbf{p}^2 + m_{\mathrm{R}}^2 \tag{2.56}$$

$$m_{\mathrm{R}}^2 = m_0^2 + \lambda_0 \times (\text{divergent integral}) + \mathrm{O}(\lambda_0^2)) \tag{2.57}$$

  similar for λ_{R}
  Reason: per unit volume: infinite number of dof

$$m_0, \lambda_0 \rightarrow m_{\mathrm{R}}, \lambda_{\mathrm{R}} \tag{2.58}$$

  involves infinite (undefined) relations

  ⟶*regularize, renormalize*

- Regularize:
  modify short distances such that theory becomes defined, in particular Feynman diagram integrals become finite, depend on some parameter. Infinities appear when parameter is removed. e.g. $a \to 0$. Most popular versions:

  1 Regularize Feynman diagram integrals,
    e.g. "dimensionally" $\int \mathrm{d}^4 p \to \int \mathrm{d}^n p\,,\ n = 4-\epsilon$

  2 Regularize the path integral: lattice with spacing a

- Renormalize:
  take a limit of some regularization parameter ($\epsilon \to 0\,,\ a \to 0$)
  at fixed $m_\mathrm{R}, \lambda_\mathrm{R}$

$$\text{observable}(m_\mathrm{R}, \lambda_\mathrm{R}) = \lim_{a\to 0} \{\text{observable}(a, m_0, \lambda_0)\}_{m_\mathrm{R},\lambda_\mathrm{R}} \tag{2.59}$$

$$m_0(a)\,, \lambda_0(a)\,, : \quad \lim_{a\to 0} \lambda_0(a) = \infty \quad \text{is possible (allowed)} \tag{2.60}$$

more precisely, everything is dimensionless, measure masses etc in units of a

$$\text{observable}(m_\mathrm{R}, \lambda_\mathrm{R}) = \lim_{am_\mathrm{R}\to 0} \{\text{observable}(am_0, \lambda_0)\}_{m_\mathrm{R},\lambda_\mathrm{R}} \tag{2.61}$$

$$am_0(am_r)\,, \lambda_0(am_r)\,, : \quad \lim_{am_r\to 0} \lambda_0(am_r) = \infty \quad \text{is possible (allowed)} \tag{2.62}$$

bare parameters = parameters in the Lagrangian are not observable, irrelevant; once they are eliminated there are no divergences.

1) Dimensional regularization
   mixes definition and and approximation
   gives <u>asymptotic</u> expansion in λ_R
   is tremendously successful (QED!)

2) Lattice regularization
   non-perturbative *definition* of the theory
   validity, precision of expansion in λ_R can be checked

3 Scalar fields on the lattice

3.1 Hypercubic lattice

lattice:

$$\Lambda = a\mathbb{Z}^4 = \{x_\mu = an_\mu |\ n_\mu \in \mathbb{Z}\,,\ \mu = 0,1,2,3\} \tag{3.1}$$

finite lattice = lattice in finite volume

$$\Lambda = \{x_\mu = an_\mu |\ n_\mu = 0,1,\ldots L_\mu/a\} \tag{3.2}$$

mostly

$$L_0 = T\,,\ L_1 = L_2 = L_3 = L\,. \tag{3.3}$$

Having the same spacing in all directions enhances the symmetry and is relevant.

Integral as for the action

$$S = \int \mathrm{d}^4x\,\mathcal{L}(x) \to a^4 \sum_x \mathcal{L}(x) \equiv a^4 \sum_{n_0,n_1,n_2,n_3} \mathcal{L}(an_0, an_1, an_2, an_3) \quad \text{with} \quad x_\mu = an_\mu\,. \tag{3.4}$$

Derivatives:

$$\text{forward:} \quad \partial_\mu\phi(x) \equiv (\partial_\mu\phi)(x) = \frac{1}{a}[\phi(x+a\hat{\mu}) - \phi(x)] \tag{3.5}$$

$$\text{backward:} \quad \partial^*_\mu\phi(x) \equiv (\partial^*_\mu\phi)(x) = \frac{1}{a}[\phi(x) - \phi(x-a\hat{\mu})] \tag{3.6}$$

$$\text{symmetric:} \quad \tilde{\partial}_\mu\phi(x) \equiv \frac{1}{2}(\partial^*_\mu + \partial_\mu)\phi(x) = \frac{1}{2a}[\phi(x+a\hat{\mu}) - \phi(x-a\hat{\mu})] \tag{3.7}$$

Partial integration

$$\int \mathrm{d}^4x\,(\partial_\mu\phi^*(x))(\partial_\mu\phi(x)) = -\int \mathrm{d}^4x\,\phi^*(x)\partial_\mu\partial_\mu\phi(x) \tag{3.8}$$

is valid
– in finite volume with pbc: $\phi(x + L_\mu\hat{\mu}) = \phi(x)$
– in infinite space-time and fields which vanish fast enough at infinity
Partial summation

$$a^4 \sum_x \psi(x)\partial_\mu\phi(x) = a^3 \sum_x \psi(x)[\phi(x+a\hat{\mu}) - \phi(x)] \tag{3.9}$$

$$= a^3[\sum_y \psi(y-a\hat{\mu})\phi(y) - \sum_y \psi(y)\phi(y)] \tag{3.10}$$

$$= -a^4 \sum_y (\partial^*_\mu\psi(y))\,\phi(y) \tag{3.11}$$

$$[\,\psi(x) = \partial_\mu\phi^*(x)\,] \tag{3.12}$$

$$a^4 \sum_x \partial_\mu\phi^*(x)\partial_\mu\phi(x) = -a^4 \sum_x \phi^*(x)\partial^*_\mu\partial_\mu\phi(x)\,. \tag{3.13}$$

This gives the action for a scalar field on the lattice

$$S = a^4 \sum_x \{\partial_\mu \phi^*(x) \partial_\mu \phi(x) + m^2 |\phi|^2(x) + \frac{\lambda_0}{4} |\phi|^4(x)\} \tag{3.14}$$

$$= a^4 \sum_x \{-\phi^*(x) \partial_\mu^* \partial_\mu \phi(x) + m^2 |\phi|^2(x) + \frac{\lambda_0}{4} |\phi|^4(x)\} \tag{3.15}$$

$$= \phi_n^* M_{nm} \phi_m = \phi^\dagger M \phi \,, \quad \phi_n = a\phi(na)\,, n = (n_0, n_1, n_2, n_3)\,, \tag{3.16}$$

$$M = M^\dagger \,, \quad M > 0 \,. \qquad \text{Check as an exercise} \tag{3.17}$$

Ordered the fields in some way ... Let us study the free field, $\lambda_0 = 0$. We expect particles, states with a definite momentum. Decoupled (free). So expect that the action is diagonalized in momentum space:

$$S = \tilde{\phi}^\dagger \tilde{M} \tilde{\phi} \,, \quad \tilde{M}_{nm} = \tilde{\mu}_m^2 \delta_{mn} \,. \tag{3.18}$$

Let's look at momentum space first.

3.2 Momentum space

Plane waves are

$$\mathrm{e}^{ipx} = \mathrm{e}^{ip_\mu x_\mu} \,. \tag{3.19}$$

Expanding a field in plane waves is a Fourrier transformation,

$$f(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{ipx} \tilde{f}(p) \,. \tag{3.20}$$

Since

$$\mathrm{e}^{ip_\mu x_\mu} = \mathrm{e}^{i(p_\mu x_\mu + 2\pi n_\mu)} = \mathrm{e}^{i(p_\mu + 2\pi/a) x_\mu} \tag{3.21}$$

the momenta

$$p_\mu \Leftrightarrow p_\mu + \frac{2\pi}{a} \tag{3.22}$$

are equivalent, and we can restrict them to the Brillouin zone,

$$-\frac{\pi}{a} \le p_\mu < \frac{\pi}{a} \,. \tag{3.23}$$

So we have

$$f(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{ipx} \tilde{f}(p) \tag{3.24}$$

in infinite volume.

Finite volume

If in addition we put ourselves into a finite volume, there are clearly also a finite number of momenta, $\prod_\mu L_\mu/a$ of them. We can then write a field as

$$f(x) = \frac{1}{V} \sum_p \mathrm{e}^{ipx} \tilde{f}(p) \,, \quad V = L_0 L_1 L_2 L_3 = T\, L_1 L_2 L_3 \,. \tag{3.25}$$

Usually one chooses periodic boundary conditions (PBC),

$$f(x+\hat{\mu}L_\mu) = \mathrm{e}^{i\theta_\mu}\, f(x) \;\rightarrow\; \sum_p \mathrm{e}^{ipx}\mathrm{e}^{ip_\mu L_\mu}\tilde{f}(p) = \sum_p \mathrm{e}^{ipx}\mathrm{e}^{i\theta_\mu}\tilde{f}(p)\,, \tag{3.26}$$

(no summation over μ) and one has

$$\mathrm{e}^{ip_\mu L_\mu} = \mathrm{e}^{i\theta_\mu} \;\rightarrow\; p_\mu = \frac{2\pi}{L_\mu}k_\mu + \frac{\theta_\mu}{L_\mu}\,, \quad k_\mu \in \{0,1,\ldots,L_\mu/a-1\} \tag{3.27}$$

(or a different range: $-\frac{L_\mu}{2a} \leq p_\mu < \frac{L_\mu}{2a}$). Normally one has

$$\theta_\mu = 0 \qquad \text{“PBC”} \tag{3.28}$$

$$\theta_\mu = \pi \qquad \text{“APBC”} \tag{3.29}$$

but general θ_μ allows more flexibility.

As the volume becomes bigger, our spacings in momentum space

$$\epsilon_\mu = \frac{2\pi}{L_\mu} \tag{3.30}$$

shrink and

$$f(x) = \frac{1}{V}\sum_p \mathrm{e}^{ipx}\tilde{f}(p) = \epsilon_0\ldots\epsilon_3 \sum_p \frac{1}{(2\pi)^4}\mathrm{e}^{ipx}\tilde{f}(p) \tag{3.31}$$

$$\rightarrow \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{ipx}\tilde{f}(p)\,. \tag{3.32}$$

In the exercise we show

$$\frac{a^4}{V}\sum_p \mathrm{e}^{ip(x-y)} = \underbrace{\prod_\mu \delta_{n_\mu m_\mu}}_{=a^4\,\delta(x-y)} \qquad [\ x_\mu = a n_\mu\ ,\ y_\mu = a m_\mu\ ] \tag{3.33}$$

$\delta(x-y)$: lattice delta-function: $a^4\sum_x \delta(x-y)\,f(x) = f(y)$ and we always identify $f(x) = f(x + a\hat{\mu}L_\mu)$.
We can take the infinite volume limit

$$\int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\mathrm{e}^{ip(x-y)} = \delta(x-y)\,. \tag{3.34}$$

Of course we also have (written in a different way):

$$\frac{a^4}{V}\sum_x \mathrm{e}^{ix(p-q)} \;=\; \prod_\mu \delta_{n_\mu m_\mu} = \delta(p-q)\prod_\mu \epsilon_\mu = \frac{(2\pi)^4}{V}\,\delta(p-q) \tag{3.35}$$

$$[\ p_\mu = n_\mu 2\pi/L_\mu\ ,\ q_\mu = m_\mu 2\pi/L_\mu\ ] \tag{3.36}$$

$$a^4\sum_x \mathrm{e}^{ix(p-q)} \;=\; (2\pi)^4\,\delta(p-q) \tag{3.37}$$

3.3 Green functions of the free field

$$S \;=\; \sum_{mn} \phi_n^* M_{nm}\phi_m = \phi^\dagger M \phi\,, \quad \phi_n = a\phi(na)\ . \tag{3.38}$$

$$M \;=\; M^\dagger\,, \quad M > 0\,, \tag{3.39}$$

so there is a unitary matrix U,

$$U^\dagger U = 1\,, \quad (U^\dagger M U)_{nm} = \delta_{mn}\mu_m^2\,, \mu_m \in \mathbb{R}. \tag{3.40}$$

To solve the free theory, we introduce a generating functional ($J, \bar{J}$ = sources)

$$Z(J,\bar{J}) \;=\; \int \mathrm{D}[\phi]\mathrm{D}[\phi^*]\,\mathrm{e}^{-S[\phi]+a^4\sum_x[\bar{J}(x)\phi(x)+\phi^*(x)J(x)]}\,, \tag{3.41}$$

$$\langle\phi(x)\phi^*(y)\rangle \;=\; Z(0,0)^{-1}a^{-8}\,\frac{\partial}{\partial\bar{J}(x)}\frac{\partial}{\partial J(y)}Z(J,\bar{J})\Big|_{J=\bar{J}=0}\,. \tag{3.42}$$

It is evaluated as

$$Z(J,\bar{J}) \;=\; \int \mathrm{D}[\phi]\mathrm{D}[\phi^*]\,\mathrm{e}^{\sum_m[-|\tilde{\phi}|_m^2\mu_m^2+\tilde{\bar{J}}_m\tilde{\phi}_m+\tilde{\phi}_m^*\tilde{J}_m]} \quad [\ \tilde{\phi}_m = U^\dagger_{mn}\phi_n = U^*_{nm}\phi_n\,,\ \ldots \tag{3.43}$$

$$\mathrm{D}[\tilde{\phi}]\mathrm{D}[\tilde{\phi}^*] = \mathrm{D}[\phi]\mathrm{D}[\phi^*]\det U\det U^\dagger = \mathrm{D}[\phi]\mathrm{D}[\phi^*]\ ] \tag{3.44}$$

$$= \prod_m \int \mathrm{d}\tilde{\phi}_m\mathrm{d}\tilde{\phi}_m^*\,\mathrm{e}^{-|\tilde{\phi}_m|^2\mu_m^2+\tilde{\bar{J}}_m\tilde{\phi}_m+\tilde{\phi}_m^*\tilde{J}_m} \tag{3.45}$$

$$= \prod_m \frac{\pi}{\mu_m^2}\,\mathrm{e}^{\tilde{\bar{J}}_m\tilde{J}_m\mu_m^{-2}} = (\det(M/\pi))^{-1}\,\mathrm{e}^{\bar{J}^T M^{-1} J} \tag{3.46}$$

$$= (\det(M/\pi))^{-1}\,\mathrm{e}^{a^8\sum_{x,y}\bar{J}(x)G(x,y)J(y)} \tag{3.47}$$

with

$$J_n = a^3 J(na), \quad M^{-1}_{nm} = a^2 G(an, am) \tag{3.48}$$

with

$$[-\partial^*_\mu \partial_\mu + m^2] G(x,y) = \delta(x-y) \tag{3.49}$$

From this we get the two-point function

$$Z(0,0)^{-1} \frac{\partial}{\partial \bar{J}(x)} \frac{\partial}{\partial J(y)} Z(J,\bar{J}) \tag{3.50}$$

$$= \frac{\partial}{\partial J(y)} a^8 [\sum_z G(x,z) J(z)\,] \times \mathrm{e}^{a^8 \sum_{z,z'} \bar{J}(z) G(z,z') J(z)} \tag{3.51}$$

$$= a^8 G(x,y)\, \mathrm{e}^{a^8 \sum_{z,z'} \bar{J}(z) G(z,z') J(z)} \tag{3.52}$$

$$+ \text{ terms that vanish when we set } J = \bar{J} = 0 \tag{3.53}$$

$$\to \langle \phi(x) \phi^*(y) \rangle = G(x,y)\,. \tag{3.54}$$

The matrix U which diagonalizes M is the matrix formed from the fourrier transformation. This is due to translation invariance (periodic boundary conditions or infinite volume) and the fact that M just contains finite differences as non-trivial terms. So we have

$$G(x,y) = G(x-y,0) = \frac{1}{V} \sum_p \mathrm{e}^{ip(x-y)}\, \tilde{G}(p) \quad \text{a graph} \tag{3.55}$$

$$\tilde{G}(p) = \frac{1}{\hat{p}^2 + m^2} \sim \frac{1}{p^2 + m^2} + \mathrm{O}(a^2\, p_\mu^2) \quad \text{exercise} \tag{3.56}$$

Exercise

What are the higher point functions, such as $G(u,v,w), G(u,v,w,x)$?

3.4 Transfer matrix

We treat here a real scalar field. The complex one is basically two copies of the real one.

The simple scalar action allows for the explicit derivation of a transfer matrix just like in quantum mechanics.

We set $a = 1$ (all dimensionful quantities in units of a until eq. (3.79)). The action is

$$S = \sum_x \{ \tfrac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + \frac{m^2}{2} \phi^2(x) + \frac{\lambda_0}{4} \phi^4(x) \} \tag{3.57}$$

The transfer matrix acts between timeslices. Therefore we collect all variables in a timeslice:

$$\Phi(x_0) = \{ \phi(x_0, \mathbf{x})\,, \mathbf{x} \in \Lambda_{\text{space}} \} \tag{3.58}$$

The action is rewritten as

$$S = \sum_{x_0}\sum_{\mathbf{x}} \tfrac{1}{2}[\phi(x_0+1,\mathbf{x})-\phi(x_0,\mathbf{x})]^2 + V(\Phi(x_0)) \tag{3.59}$$

$$V(\Phi(x_0)) = \sum_{\mathbf{x}} \{\tfrac{1}{2}\partial_j\phi(x)\partial_j\phi(x) + \frac{m^2}{2}\phi^2(x) + \frac{\lambda_0}{4}\phi^4(x)\}\,. \tag{3.60}$$

We want to show as in QM

$$Z = \operatorname{Tr} \mathrm{e}^{-T\hat{H}} = \int \mathrm{D}[\phi]\,\mathrm{e}^{-S}\,. \tag{3.61}$$

First we need a Hilbert space and operators. Introduce $\phi(\mathbf{x})$ as operator $\hat{\phi}(\mathbf{x})$ and canonical conjugate $\hat{\pi}(\mathbf{x})$ (Schrödinger picture):

$$[\hat{\phi}(\mathbf{x}),\hat{\phi}(\mathbf{y})] = 0 = [\hat{\pi}(\mathbf{x}),\hat{\pi}(\mathbf{y})] \tag{3.62}$$

$$[\hat{\phi}(\mathbf{x}),\hat{\pi}(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}) \equiv \prod_i \delta_{x_i y_i} \tag{3.63}$$

and just as usually (think of $\phi(\mathbf{x}) = q, \pi(\mathbf{x}) = p$)

$$\langle\phi'(\mathbf{x})|\phi(\mathbf{x})\rangle = \delta(\phi'(\mathbf{x})-\phi(\mathbf{x})) \tag{3.64}$$

$$\langle\pi'(\mathbf{x})|\pi(\mathbf{x})\rangle = \delta(\pi'(\mathbf{x})-\pi(\mathbf{x})) \tag{3.65}$$

$$\langle\phi'(\mathbf{x})|F(\hat{\phi}(\mathbf{x}))|\phi(\mathbf{x})\rangle = \delta(\phi'(\mathbf{x})-\phi(\mathbf{x}))F(\phi(\mathbf{x})) \tag{3.66}$$

$$\langle\phi(\mathbf{x})|\pi(\mathbf{x})\rangle = \frac{1}{(2\pi)^{1/2}}\mathrm{e}^{-i\phi(\mathbf{x})\pi(\mathbf{x})}\,. \tag{3.67}$$

The total Hilbert space is the direct product of Hilbert spaces at each $\mathbf{x}$

$$|\Phi\rangle = \prod_{\mathbf{x}} |\phi(\mathbf{x})\rangle = |\phi((0,0,0))\rangle|\phi(a,0,0)\rangle\ldots|\phi((L-1,L-1,L-1))\rangle\,. \tag{3.68}$$

An explicit representation is the Schrödinger representation:

$$\psi[u] = \psi[\{u(\mathbf{x})\}]\,,\ \langle\psi|\psi\rangle = \int\prod_{\mathbf{x}}\mathrm{d}u(\mathbf{x})\,|\psi[u]|^2 \tag{3.69}$$

$$\hat{\phi}(\mathbf{x})\,\psi[u] = u(\mathbf{x})\,\psi[u] \tag{3.70}$$

$$\hat{\pi}(\mathbf{x})\,\psi[u] = -i\,\frac{\partial}{\partial u(\mathbf{x})}\,\psi[u]\,. \tag{3.71}$$

We factorize the "Boltzmann factor"

$$\mathrm{e}^{-S} = \prod_{x_0} \mathrm{e}^{-\frac{1}{2}V(\Phi(x_0+1))}\,\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{x}}[\phi(x_0+1,\mathbf{x})-\phi(x_0,\mathbf{x})]^2}\,\mathrm{e}^{-\frac{1}{2}V(\Phi(x_0))} \tag{3.72}$$

Now consider the difficult term (as in quantum mechanics):

$$\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{x}}[\phi(x_0+1,\mathbf{x})-\phi(x_0,\mathbf{x})]^2} = \prod_{\mathbf{x}} \mathrm{e}^{-\frac{1}{2}[\phi(x_0+1,\mathbf{x})-\phi(x_0,\mathbf{x})]^2} \tag{3.73}$$

$$= \prod_{\mathbf{x}} \langle\phi(x_0+1,\mathbf{x})|\mathrm{e}^{-\frac{1}{2}\hat{\pi}(\mathbf{x})^2}|\phi(x_0,\mathbf{x})\rangle \tag{3.74}$$

$$= \langle\Phi(x_0+1)|\mathrm{e}^{-\frac{1}{2}\sum_{\mathbf{x}}\hat{\pi}(\mathbf{x})^2}|\Phi(x_0)\rangle \tag{3.75}$$

and then

$$e^{-\frac{1}{2}V(\Phi(x_0+1))}\, e^{-\frac{1}{2}\sum_{\mathbf{x}}[\phi(x_0+1,\mathbf{x})-\phi(x_0,\mathbf{x})]^2}\, e^{-\frac{1}{2}V(\Phi(x_0))} \tag{3.76}$$

$$= \langle\Phi(x_0+1)|e^{-\frac{1}{2}V(\hat\Phi)}\, e^{-\frac{1}{2}\sum_{\mathbf{x}}\hat\pi(\mathbf{x})^2}\, e^{-\frac{1}{2}V(\hat\Phi)}|\Phi(x_0)\rangle\,. \tag{3.77}$$

The TM,

$$\hat{\mathbb{T}} = e^{-\frac{1}{2}V(\hat\Phi)}\, e^{-\frac{1}{2}\sum_{\mathbf{x}}\hat\pi(\mathbf{x})^2}\, e^{-\frac{1}{2}V(\hat\Phi)}\,. \tag{3.78}$$

is hermitian and positive. Therefore (restoring a) writing

$$\hat{\mathbb{T}} = e^{-a\hat H}\,, \tag{3.79}$$

makes sense. It defines the lattice hamiltonian $\hat H$, a hermitian operator.

$$\hat H = \hat H^\dagger\,,\ \hat H \geq 0\,. \tag{3.80}$$

A formal expansion in a gives:

$$\hat H = \sum_{\mathbf{x}}\{\tfrac{1}{2}\hat\pi(\mathbf{x})^2 + \tfrac{1}{2}\partial_j\hat\phi(\mathbf{x})\partial_j\hat\phi(\mathbf{x}) + \frac{m^2}{2}\hat\phi^2(\mathbf{x}) + \frac{\lambda_0}{4}\hat\phi^4(\mathbf{x})\} + \mathrm{O}(a^2)\,. \tag{3.81}$$

Remarks:

- The exisitence of a positive, hermitian operator is called positivity. It corresponds to unitarity in Minkowsky space, the conservation of probablity, very important.
- An action with just 2nd order derivatives was important in deriving $\hat{\mathbb{T}}$.
- We emphasize the huge Hilbert space. A QM degree of freedom at each space-point.

3.5 Translation operator, spectral representation

Let us introduce the spatial translation operator by

$$\hat U(\mathbf{x})|\psi\rangle = |\psi'\rangle\,, \tag{3.82}$$

$$\langle\psi_1'|\hat\phi(\mathbf{y})|\psi_2'\rangle = \langle\psi_1|\hat\phi(\mathbf{y}-\mathbf{x})|\psi_2\rangle\,. \tag{3.83}$$

This means that

$$\langle\psi_1|\hat U(\mathbf{x})^\dagger\hat\phi(\mathbf{y})\hat U(\mathbf{x})|\psi_2\rangle = \langle\psi_1|\hat\phi(\mathbf{y}-\mathbf{x})|\psi_2\rangle \tag{3.84}$$

and therefore the operators transform as

$$\hat U(\mathbf{x})^\dagger\hat\phi(\mathbf{y})\hat U(\mathbf{x}) = \hat\phi(\mathbf{y}-\mathbf{x})\,. \tag{3.85}$$

Clearly our Hamiltonian is chosen invariant

$$\hat U(\mathbf{x})^\dagger\hat H\hat U(\mathbf{x}) = \hat H\,. \tag{3.86}$$

Another property that we need is that $\hat{U}(\mathbf{x})$ is unitary. We look at the Schrödinger representation

$$\langle\psi_1|\hat{\phi}(\mathbf{y}-\mathbf{x})|\psi_2\rangle = \int\prod_{\mathbf{z}} \mathrm{d}v(\mathbf{z})\,\psi_1^*[v]v(\mathbf{y}-\mathbf{x})\psi_2[v] \tag{3.87}$$

$$= \int\prod_{\mathbf{z}} \mathrm{d}v'(\mathbf{z})\,\psi_1^*[v']\,v'(\mathbf{y})\,\psi_2[v']\,, \quad [\,v'(\mathbf{y}) = v(\mathbf{y}+\mathbf{x})\,] \tag{3.88}$$

$$= \int\prod_{\mathbf{z}} \mathrm{d}v(\mathbf{z})\,(\psi_1')^*[v]\,v(\mathbf{y})\,\psi_2'[v] \;=\; \langle\psi_1'|\hat{\phi}(\mathbf{y})|\psi_2'\rangle \tag{3.89}$$

$$\psi_i'[v] = \hat{U}(\mathbf{x})\psi_i[v] = \psi_i[v']\,,\; v'(\mathbf{y}) = v(\mathbf{y}+\mathbf{x}) \tag{3.90}$$

From this we see immediately that

$$\langle\psi_1'|\psi_2'\rangle = \langle\psi_1|\hat{U}(\mathbf{x})^\dagger\hat{U}(\mathbf{x})|\psi_2\rangle = \langle\psi_1|\psi_2\rangle\,. \tag{3.91}$$

There are simultaneous eigenstates of $\hat{H}$ and $\hat{U}$. For eigenstates of $\hat{U}$ we have

$$\hat{U}(\mathbf{x})|\lambda(\mathbf{x})\rangle = \lambda(\mathbf{x})|\lambda(\mathbf{x})\rangle\,, \quad |\lambda(\mathbf{x})|^2 = 1\,, \quad \lambda(\mathbf{x}) = \mathrm{e}^{i\alpha(\mathbf{x})} \tag{3.92}$$

$$\alpha(\mathbf{x})+\alpha(\mathbf{y}) = \alpha(\mathbf{x}+\mathbf{y}) \;\rightarrow\; \alpha(\mathbf{x}) = -\mathbf{x}\mathbf{p} \quad \text{with some} \quad \mathbf{p}\,. \tag{3.93}$$

The last equation is just because of the linearity seen before. So we have with $|\lambda(\mathbf{x})\rangle = |\mathbf{p},n\rangle$, n for other quantum numbers,

$$\hat{U}(\mathbf{x})|\mathbf{p},n\rangle = \mathrm{e}^{-i\mathbf{x}\mathbf{p}}|\mathbf{p},n\rangle\,, \tag{3.94}$$

$$\hat{H}|\mathbf{p},n\rangle = (E(\mathbf{p},n)+E_0)|\mathbf{p},n\rangle\,, \tag{3.95}$$

$$E_0 \;:\; \text{the ground state energy} \tag{3.96}$$

The physical interpretation is that $\mathbf{p}$ is the momentum of the state. And on a finite, periodic lattice we have the restrictions for $\mathbf{p}$ as before. As normalization we choose

$$\langle\mathbf{p},n|\mathbf{p}',n'\rangle = 2\,E(\mathbf{p},n)\,L^3\,\delta(\mathbf{p}-\mathbf{p}')\,\delta_{nn'} \tag{3.97}$$

Let us now look at the 2-point function and use translation invariance to derive the spectral representation.

$$T\to\infty \quad , \quad L \text{ finite} \tag{3.98}$$

$$G(x-y) = \langle\phi(x)\,\phi(y)\rangle \tag{3.99}$$

$$= \langle 0|\hat{\phi}(\mathbf{x})\mathrm{e}^{-\hat{H}\,|x_0-y_0|}\hat{\phi}(\mathbf{y})|0\rangle\,\mathrm{e}^{E_0\,(x_0-y_0)} \tag{3.100}$$

$$= \frac{1}{L^3}\sum_{\mathbf{p},n}\frac{1}{2\,E(\mathbf{p},n)}\,\mathrm{e}^{-E(\mathbf{p},n)\,|x_0-y_0|}\langle 0|\hat{\phi}(\mathbf{x})|\mathbf{p},n\rangle\langle\mathbf{p},n|\hat{\phi}(\mathbf{y})|0\rangle$$

and, assuming the translation invariance of the ground state, $\hat{U}^\dagger(\mathbf{x})|0\rangle = |0\rangle$ (for a finite system it can be proven that the ground state is translation invariant)

$$\langle 0|\hat{\phi}(\mathbf{x})|\mathbf{p},n\rangle = \langle 0|\hat{U}(\mathbf{x})\hat{\phi}(0)\hat{U}^\dagger(\mathbf{x})|\mathbf{p},n\rangle \tag{3.101}$$

$$= \mathrm{e}^{i\mathbf{p}\mathbf{x}}\langle 0|\hat{\phi}(0)|\mathbf{p},n\rangle \tag{3.102}$$

$$\langle 0|\hat{\phi}(\mathbf{y})|\mathbf{p},n\rangle\langle\mathbf{p},n|\hat{\phi}(\mathbf{x})|0\rangle = \mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}|\langle 0|\hat{\phi}(0)|\mathbf{p},n\rangle|^2 \tag{3.103}$$

This gives

$$G(x-y) = \frac{1}{L^3}\sum_{\mathbf{p},n} \mathrm{e}^{E(\mathbf{p},n)\,|x_0-y_0|}\,\mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}\,\frac{|\langle 0|\hat{\phi}(0)|\mathbf{p},n\rangle|^2}{2E(\mathbf{p},n)} \tag{3.104}$$

$$= \frac{1}{L^3}\int \mathrm{d}\omega \sum_{\mathbf{p}} \rho_{\mathrm{L}}(\omega,\mathbf{p})\,\mathrm{e}^{-\omega\,|x_0-y_0|}\,\mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}\,, \tag{3.105}$$

$$\rho_{\mathrm{L}}(\omega,\mathbf{p}) = \sum_n \frac{|\langle 0|\hat{\phi}(0)|\mathbf{p},n\rangle|^2}{2E(\mathbf{p},n)}\,\delta(\omega-(E(\mathbf{p},n)))\,. \tag{3.106}$$

And in infinite volume

$$G(x-y) \to \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\int \mathrm{d}\omega\,\rho(\omega,\mathbf{p})\;\mathrm{e}^{-\omega\,|x_0-y_0|}\,\mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}\,, \tag{3.107}$$

$$\rho(\omega,\mathbf{p}) = \lim_{L\to\infty}\rho_{\mathrm{L}}(\omega,\mathbf{p}) \quad = \text{spectral density.} \tag{3.108}$$

From our derivation we have seen that the spectral density is nothing but the coupling of a field-operator to states of definite momentum: $|\langle 0|\hat{\phi}(0)|\mathbf{p},n\rangle|^2$. The x-dependence of the two-point function follows from space and time translations, in terms of two dynamical quantities, $E(\mathbf{p},n)\,,\ \rho(\omega,\mathbf{p})$.

3.6 Timeslice correlation function and spectrum of the free thory

The free propagator is

$$G(x) = \langle\phi(x)\phi(0)\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{ipx}\tilde{G}(p) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{i\mathbf{px}} G(x_0;\mathbf{p}) \tag{3.109}$$

$$G(x_0;\mathbf{p}) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}p_0}{(2\pi)} \mathrm{e}^{-ip_0x_0}\, G(p)\,. \tag{3.110}$$

From the above use of translation invariance we have

$$G(x_0;\mathbf{p}) = \int \mathrm{d}\omega\, \rho(\omega,\mathbf{p})\, \mathrm{e}^{-\omega|x_0|} = \sum_n c_n^2(\mathbf{p})\, \mathrm{e}^{-E(\mathbf{p},n)|x_0|}\,. \tag{3.111}$$

We now evaluate this explicitly.

$$G(x_0;\mathbf{p}) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}p_0}{(2\pi)} \frac{\mathrm{e}^{ip_0x_0}}{\hat{p}^2+m^2} \tag{3.112}$$

$$\text{for } x_0 \le 0 \qquad [\ \phi = a\tilde{p}_0\,, \quad n_0 = -x_0/a\ ] \tag{3.113}$$

$$= a\int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{(2\pi)} \mathrm{e}^{i\phi n_0} \frac{1}{a^2m^2 + a^2\hat{\mathbf{p}}^2 + 2(1-\cos\phi)} \tag{3.114}$$

$$= a\int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{(2\pi)} \mathrm{e}^{i\phi n_0} \frac{1}{A - 2\cos\phi} \qquad [\ A = 2 + a^2m^2 + a^2\hat{\mathbf{p}}^2\ ] \tag{3.115}$$

$$[\ z = \mathrm{e}^{i\phi}\,,\ \mathrm{d}z = iz\mathrm{d}\phi\,, 2\cos\phi = (z+z^{-1})\ ] \tag{3.116}$$

$$= a\frac{1}{2\pi i}\oint_{|z|=1} \mathrm{d}z\, z^{n_0} \frac{1}{z[A-(z+z^{-1})]} = \sum \text{Residues} \tag{3.117}$$

$$\tag{3.118}$$

The poles are at ($\omega > 0$)

$$D = z[A-(z+z^{-1})] = 0\,,\ z_1 = \mathrm{e}^{-a\omega}\,,\ z_2 = \mathrm{e}^{a\omega}\,, \tag{3.119}$$

$$A = 2\cosh(a\omega)\ \ \omega > 0\,, \tag{3.120}$$

$$\to\ D = -(z-\mathrm{e}^{-a\omega})(z-\mathrm{e}^{a\omega})\,. \tag{3.121}$$

Only z_1 is inside the circle. Its residue is

$$\frac{\mathrm{e}^{-n_0 a\omega}}{2\sinh(a\omega)}\,. \tag{3.122}$$

So we have

$$G(x_0;\mathbf{p}) = \frac{\mathrm{e}^{-|x_0|\omega}}{2\sinh(a\omega)/a}\,, \quad \text{(also for } x_0 > 0) \tag{3.123}$$

$$\to\ E(\mathbf{p}) = \omega(\mathbf{p})\,, 2[\cosh(aE(\mathbf{p}))-1] = a^2m^2 + a^2\hat{\mathbf{p}}^2\,, \tag{3.124}$$

$$\text{spectral density:} \qquad \rho(\omega,\mathbf{p}) = a\delta(\omega - E(\mathbf{p}))\frac{1}{2\sinh(a\omega)} \tag{3.125}$$

We observe

- There is only one intermediate state per $\mathbf{p}$; the spectral density is a single δ-function. Such a state is created from the vacuum by

$$\hat{O}_1(\mathbf{p}) = C a^3 \sum_{\mathbf{x}} \hat{\phi}(\mathbf{x}) \mathrm{e}^{i\mathbf{p}\mathbf{x}} \,. \tag{3.126}$$

  Namely we had

$$G(x_0 - y_0; \mathbf{p}) \propto \langle O_1(x_0, -\mathbf{p}) O_1(y_0, \mathbf{p}) \rangle \propto \langle 0 | O_1(\mathbf{p})^\dagger \mathrm{e}^{-|x_0 - y_0| \hat{H}} O_1(\mathbf{p}) | 0 \rangle \tag{3.127}$$

- The energy momentum relation is (when $a^2 \, p_i^2 \ll 1 \,, \quad a^2 m^2 \ll 1$):

$$E^2 = \mathbf{p}^2 + m^2 + \mathrm{O}(a^2) \tag{3.128}$$

$\Rightarrow$ a free particle with relativistic energy momentum relation.

- 2-particle states are simply created by

$$\hat{O}_2(\mathbf{p}, \mathbf{q}) = \hat{O}_1(\mathbf{p}) \, \hat{O}_1(\mathbf{q}) = \hat{O}_1(\mathbf{q}) \, \hat{O}_1(\mathbf{p}) \tag{3.129}$$

  A correlation function is

$$G_4 = a^{12} \sum_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}} \mathrm{e}^{-i(\mathbf{pz} + \mathbf{qw} - \mathbf{px} - \mathbf{qy})} \langle \phi(t, \mathbf{z}) \, \phi(t, \mathbf{w}) \phi(0, \mathbf{x}) \phi(0, \mathbf{y}) \rangle \tag{3.130}$$

  this gives: **a graph**

$$\text{time dependence } C_1(\mathbf{p}, \mathbf{q}) \, \mathrm{e}^{-|t|(E(\mathbf{p}) + E(\mathbf{q}))} + C_2(\mathbf{p}, \mathbf{q}) \, \delta(\mathbf{p} + \mathbf{q}) \tag{3.131}$$

  So

$$E = E(\mathbf{p}) + E(\mathbf{q}) \geq 2m \qquad \textbf{a graph} \tag{3.132}$$

For later use we note that above we have shown that

$$\mathrm{e}^{-|x_0|\omega} = \int \frac{\mathrm{d}p_0}{2\pi} \, \mathrm{e}^{ip_0 x_0} \, \frac{\sinh(a\omega)}{\cosh(a\omega) - \cos(ap_0)} \,. \tag{3.133}$$

3.7 Lattice artifacts

Let us include the $\mathrm{O}(a^2)$ effects:

$$\frac{2}{a^2}[\cosh(aE(\mathbf{p})) - 1] = E^2 + \frac{2}{4!} a^2 E^4 + \mathrm{O}(a^4) \tag{3.134}$$

$$= E^2 + \frac{(m^2 + \mathbf{p}^2)^2 a^2}{12} + \mathrm{O}(a^4) \tag{3.135}$$

$$\hat{p}_j^2 = \frac{2}{a^2}(1 - \cos p_j a) = p_j^2 - \frac{p_j^4 a^2}{12} + \mathrm{O}(a^4) \,, \tag{3.136}$$

$$E^2(\mathbf{p}) = m^2 + \mathbf{p}^2 - \underbrace{\frac{a^2}{12}[(m^2 + \mathbf{p}^2)^2 + \sum_j p_j^4]}_{\approx 10\% \text{ when } aE(\mathbf{p}) = 1} + \mathrm{O}(a^4) \,. \tag{3.137}$$

Now we have to be careful, however. This includes the mass in the Lagrangian, not an observable. We should renormalize first, even at tree level! Not unique, but very natural:

$$\begin{aligned}
\text{mass} &= \text{energy at rest}: \quad \textbf{renormalization condition} && (3.138)\\
m_{\mathrm{R}}^2 &= E^2(\mathbf{p}=0) = m^2\,(1-\frac{a^2}{12}m^2+\ldots) && (3.139)\\
m^2 &= m_{\mathrm{R}}^2\,(1+\frac{a^2}{12}m_{\mathrm{R}}^2) + \mathrm{O}(a^4) && (3.140)\\
&\to && (3.141)\\
E^2(\mathbf{p}) &= m_{\mathrm{R}}^2 + \mathbf{p}^2 \underbrace{-\frac{a^2}{12}[2m_{\mathrm{R}}^2\mathbf{p}^2 + (\mathbf{p}^2)^2 + \sum_j p_j^4]}_{\approx 10\% \text{ when } aE(\mathbf{p})=1} + \mathrm{O}(a^4)\,. && (3.142)
\end{aligned}$$

See Fig. 5, left.

Can this be improved? Better discretization?

3.8 Improvement

In general there is Symanzik improvement:

$$S \to S_{\mathrm{impr}} = S + \delta S\,, \quad \delta S = a^4 \sum_x \sum_i c_i a^{d_O - 4} O_i(x)\,, \tag{3.143}$$

Here we have

$$\begin{aligned}
\sum_i c_i a^{d_O-4} O_i(x) &= c_1\,\frac{a^2}{2}\sum_{\mu=0}^{3}[\partial_\mu\partial_\mu\phi(x)]^2 && (3.144)\\
&\neq [\sum_{\mu=0}^{3}\partial_\mu\partial_\mu\phi(x)\,]^2 && (3.145)\\
\to \quad E^2(\mathbf{p}) &= m^2 + \mathbf{p}^2 + a^2[c-\tfrac{1}{12}][(m^2+\mathbf{p}^2)^2 + \sum_j p_j^4] + \mathrm{O}(a^4) && (3.146)
\end{aligned}$$

Remarks (here not really explained):

- In general in a scalar theory in 4 dimensions: $O_i(x)$ local fields with mass dimension 6
  Even more generally: local fields with mass dimension ≥ 5

$$c_i = c_i(\lambda_0) \tag{3.147}$$

- On-shell improvement: terms which vanish by the e.o.m can be dropped (see exercise for the e.o.m.)

- Also fields in correlation functions have to be improved (correction terms). We here looked only at energies (which are on-shell).

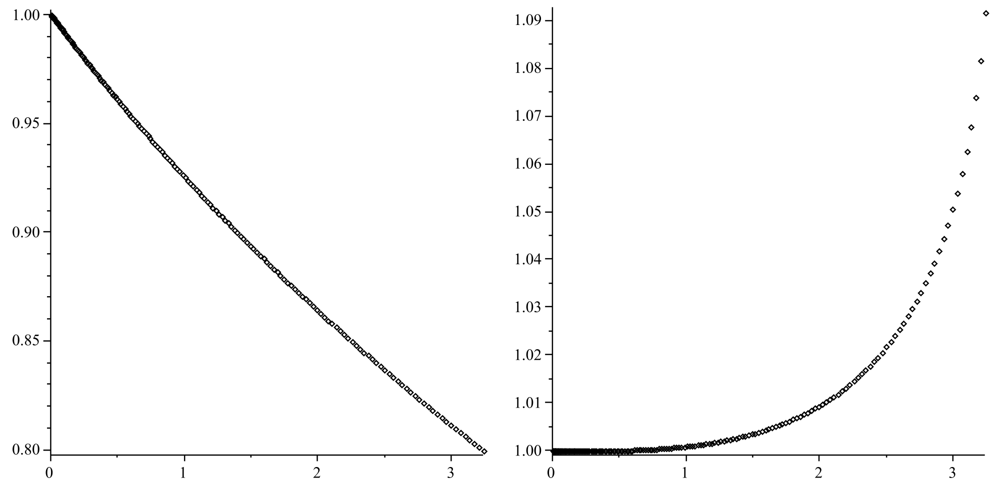


Figure 6: $E^2(\mathbf{p})/E^2(\mathbf{p})_{cont}$ *for* $\mathbf{p} = (p, 0, 0)$, $m = 0$, *against the square lattice spacing* a^2E^2. *Left side:* $c = 0$, *right side:* $c = 1/12$.

3.9 Universality

$$
\begin{aligned}
Z &= \int \mathrm{D}[\phi]\, \mathrm{e}^{-S[\phi]}\,, && (3.148) \\
\langle \phi(x)\phi(y) \rangle &= Z^{-1} \int \mathrm{D}[\phi]\, \mathrm{e}^{-S[\phi]} \phi(x)\phi(y)\,. && (3.149)
\end{aligned}
$$

$$
a^3 \sum_{\mathbf{x}} \langle \phi(x)\phi(0) \rangle \overset{x_0 \to \infty}{\sim} \mathrm{e}^{-E(\mathbf{p}=0)x_0} = \mathrm{e}^{-m_{\mathrm{R}} x_0} = \mathrm{e}^{-n_0/\xi} \qquad (3.150)
$$

A change of notation

$$
\begin{aligned}
S &\to H/(kT_{temp}) && (3.151) \\
d = 3 + 1 &\to d = 4 \quad (+0: \text{ static}) && (3.152) \\
(am_{\mathrm{R}})^{-1} &\to \xi && (3.153) \\
a \to 0 &\to \xi \to \infty && (3.154)
\end{aligned}
$$

shows that a lattice field theory in $3+1$ dimensions is a statistical model in 4 dimensions. The continuum limit is reached at a critical point. Statistical models are known to have *universality* there.

Universality means that a change of the Hamilton function which does not change the symmetries (axis permutations, $\phi \to -\phi$ etc.) gives the same correlation functions.

In the QFT this means the continuum limit is unique; it does not depend on the details of the discretization (adding e.g. $(\partial_\mu \partial_\mu \phi)^2$ term).

More precisely: the continuum limit only depends on the coefficients of the renormalizable interaction terms: $[O(\phi(x), \partial_\mu \phi(x)] \leq 4$. Eg. the addition of ϕ^6 changes only the cutoff effects.

In terms of continuum field theory best looked at in the corresponding action:

$$
S = a^4 \sum_x \{ \tfrac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + \frac{m^2}{2} \phi^2(x) \} + S_{\mathrm{int}} \qquad (3.155)
$$

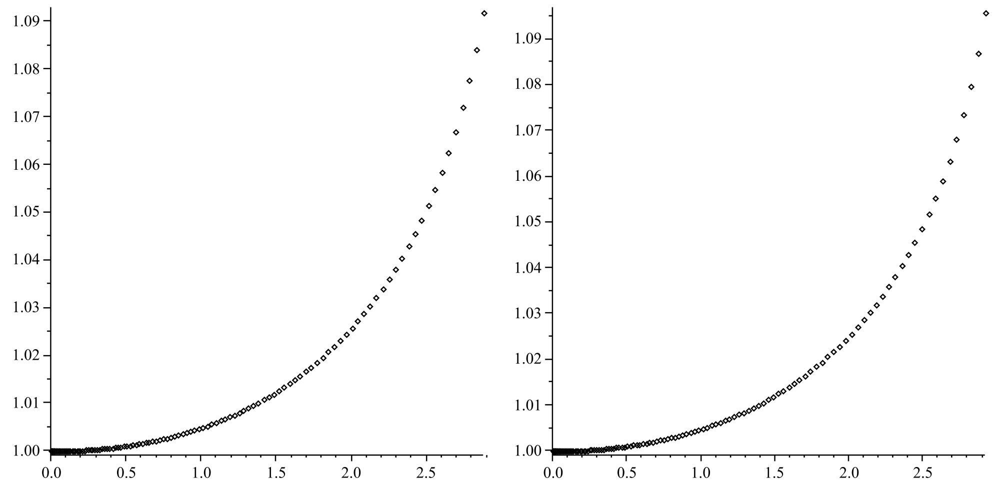


Figure 7: $E^2(\mathbf{p})/E^2(\mathbf{p})_{cont}$ *against the square lattice spacing* a^2E^2. *Improved case:* $c = 1/12$. *Left side: for* $\mathbf{p} = (p, p, 0), m = 0$, *right side:* $\mathbf{p} = (p, 0, 0), m_{\rm R} = p$.

then

- assume locality: $S_{\rm int} = a^4 \sum_x const.\, M^{-(n+4k-4)}\, \phi^n(x)(\partial_\mu\phi(x)\partial_\mu\phi(x))^k + \ldots$
- renormalizability when (we just state this):

$$S_{\rm int} = a^4 \sum_x \mathcal{L}_{\rm int}(x)\,, \quad [\mathcal{L}_{\rm int}(x)] \leq 4 \tag{3.156}$$

$$.[\Phi] = \text{mass dimension:} \quad [\partial_\mu] = 1\,,\ [\phi(x)] = 1\,, \tag{3.157}$$

and no odd powers of ϕ and Euclidean invariance

$$\mathcal{L}_{\rm int}(x) = \frac{\lambda_0}{4}\phi^4(x) \quad \rightarrow \quad \hat{H}_{\rm int} = \frac{\lambda_0}{4}\hat{\phi}^4(\mathbf{x}) \tag{3.158}$$

λ_0 is the only possible coupling constant, free parameter
QFT's are very predictive

- how well is this established (proven)?
to all order in $\lambda_0 \rightarrow \lambda_R$ (T. Reisz, lattice power counting theorem)
non-renormalizability of eg. $\phi^6(x)/m_6^2$ in the same way to all orders in m_6^{-2} .
non-perturbatively: numerical investigations are in agreement with this.

Remark: do not confuse higher dimensional operators in Symanzik improvement and a theory with higher dimensional operators in the continuum limt. In one case the coefficients are proportional to a^{d_O-4}, in the other case they are proportional to $1/m^{d_O-4}$.

4 Gauge fields on the lattice

4.1 Color, parallel transport, gauge invariance

Quarks carry color, $A, B = 1 \dots 3$ ($1 \dots N$ in an SU(N) gauge theory):

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \in \mathbb{C}^3 . \tag{4.1}$$

The basic principle is gauge invariance: different colors are completely equivalent. One can rotate the fields and physics does not change.

$$\Lambda(x) \in \mathrm{SU}(3) \; : \psi(x) \to \psi^\Lambda(x) = \Lambda(x)\, \psi(x) . \tag{4.2}$$

(There is also the possibility to rotate by a phase (U(1)), but this corresponds to electrodynamics, a separate issue). $\Lambda(x) \in \mathrm{SU}(3)$ is an arbitrary function of x. So it makes no sense to compare $\psi(x)$ and $\psi(y)$. Before comparing we have to parallel-transport $\psi(y)$, such that it transforms as $\psi(x)$. Parallel transporter:

$$P(x \leftarrow y): \quad P(x \leftarrow y) \;\to\; P^\Lambda(x \leftarrow y) = \Lambda(x)\, P(x \leftarrow y)\, \Lambda^{-1}(y) \tag{4.3}$$

$$P(x \leftarrow y)\, \psi(y) \;\to\; \Lambda(x)\, P(x \leftarrow y)\, \psi(y) . \tag{4.4}$$

The parallel transporter will in general depend on the path from y to x. Think of a straight path for definiteness. For the definition of a (continuum derivative) we need the transporter by an infinitessimal distance:

$$D_\mu \psi(x) = (D_\mu \psi)(x) = \lim_{\epsilon \to 0} [P(x \leftarrow x + \epsilon\hat{\mu})\, \psi(x + \epsilon\hat{\mu}) - \psi(x)] . \tag{4.5}$$

Every SU(N) matrix can be written in the form

$$P = \mathrm{e}^B , \quad B = -B^\dagger = \sum_{a=1}^{N^2-1} B^a T^a \tag{4.6}$$

$$B^a \in \mathbb{R} , \; T^a = -(T^a)^\dagger , \;\; \mathrm{tr}\, T^a = 0 , \;\; \mathrm{tr}\, T^a T^b = -\tfrac{1}{2}\delta_{ab} , [T^a, T^b] = f^{abc} T^c . \tag{4.7}$$

For example in SU(2):

$$T^a = \frac{1}{2i} \tau^a , \;\; \text{Pauli matrices} . \tag{4.8}$$

So we can write for an infinitessimal path

$$P(x \leftarrow x + \epsilon\hat{\mu}) = \mathrm{e}^{\epsilon A_\mu(x)} \quad \to \quad D_\mu = \partial_\mu + A_\mu . \tag{4.9}$$

A_μ has mass dimension one. If quarks were scalars, gauge invariance would force the kinetic term in the Lagrangian to look like

$$\sum_A |D_\mu \psi_A|^2 . \tag{4.10}$$

It contains a coupling to the field A_μ (gluon field), an interaction term. What about a kinetic term for A_μ? Gauge invariance is the basic principle. First in the continuum:

The basic object is D_μ because it is gauge covariant,

$$D_\mu^\Lambda = \Lambda(x) D_\mu \Lambda^{-1}(x)\,. \tag{4.11}$$

We want something Euclidean invariant and gauge invariant:

$$\begin{aligned}
& \operatorname{tr} D_\mu D_\mu = \operatorname{tr}(\partial_\mu\partial_\mu + \ldots) \quad \partial_\mu \text{ acting on what?} && (4.12)\\
& \operatorname{tr} F_{\mu\nu} F_{\mu\nu}\,, && (4.13)\\
F_{\mu\nu} &= [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] && (4.14)\\
&= T^a\{\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c\} && (4.15)\\
&= T^a F_{\mu\nu}^a\,. && (4.16)
\end{aligned}$$

Other terms have higher dimension, are not renormalizable. So the action is

$$S_{\mathrm{G}} = -\underbrace{\frac{1}{2g_0^2}}_{\text{convention}} \int \mathrm{d}^4x \, \operatorname{tr} F_{\mu\nu}(x) F_{\mu\nu}(x) = \frac{1}{2g_0^2}\int \mathrm{d}^4x\, F^2\,. \tag{4.17}$$

Unlike QED, there are interaction terms already in here even without the quark fields.

On the lattice, the quark fields are at the lattice points x, but the gluon field has to provide the parallel transporter from point to point, it is sitting on a link

$$\begin{aligned}
P(x \leftarrow x + a\hat\mu) &= U(x,\mu) \quad : \quad \longleftarrow && (4.18)\\
P(x + a\hat\mu \leftarrow x) &= U^{-1}(x,\mu) \quad : \quad \longrightarrow\,. && (4.19)
\end{aligned}$$

Gauge covariant derivative:

$$\begin{aligned}
\text{forward:} \quad D_\mu\psi(x) &= \frac{1}{a}[U(x,\mu)\psi(x+a\hat\mu) - \psi(x)] && (4.20)\\
\text{backward:} \quad D_\mu^*\psi(x) &= \frac{1}{a}[\psi(x) - U^{-1}(x - a\hat\mu,\mu)\psi(x - a\hat\mu)] && (4.21)
\end{aligned}$$

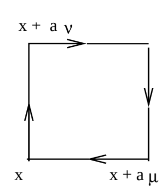
With these we could build a kinetic term for scalar quarks

$$S_\phi = a^4 \sum_x \sum_A |D_\mu \psi_A|^2 = a^4 \sum_x \sum_A \psi_A^*(-D_\mu^* D_\mu \psi)_A = a^4 \sum_x \psi^\dagger(-D_\mu^* D_\mu)\psi\,. \tag{4.22}$$

For fermions we do that later.

For the kinetic term we need a local object, gauge invariant. The most local one is the parallel-transporter around a plaquette (elementary square):

$$O_{\mu\nu}(x) = \operatorname{tr} U(x,\mu) U(x+a\hat\mu,\nu) U^{-1}(x+a\hat\nu,\mu) U^{-1}(x,\nu) \;: \tag{4.23}$$



Assume that we have a smooth classical field, so the parallel transporters are close to one, we can then write

$$U(x, \mu) = \mathrm{e}^{aA_\mu(x)} \tag{4.24}$$

Then we can expand in a (classical continuum limit)

$$\begin{aligned}
O_{\mu\nu}(x) &= \mathrm{tr}\; U(x,\mu)U(x+a\hat{\mu},\nu)U^{-1}(x+a\hat{\nu},\mu)U^{-1}(x,\nu) && (4.25)\\
&= \mathrm{tr}\; \underbrace{\mathrm{e}^{aA_\mu(x)}\mathrm{e}^{aA_\nu(x+a\hat{\mu})}}_{\mathrm{e}^B}\; \underbrace{\mathrm{e}^{-aA_\mu(x+a\hat{\nu})}\mathrm{e}^{-aA_\nu(x)}}_{\mathrm{e}^C}, && (4.26)\\
B &= a(A_\mu(x) + A_\nu(x+a\hat{\mu})) + a^2\tfrac{1}{2}[A_\mu(x)\,,\, A_\nu(x+a\hat{\mu})] + \mathrm{O}(a^3)\,, && (4.27)\\
C &= a(-A_\mu(x+a\hat{\nu}) - A_\nu(x) + a^2\tfrac{1}{2}[A_\mu(x+a\hat{\nu})\,,\, A_\nu(x)] + \mathrm{O}(a^3)\,, && (4.28)
\end{aligned}$$

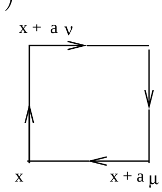
$$\begin{aligned}
\mathrm{e}^B\mathrm{e}^C &= \mathrm{e}^{B+C+\frac{1}{2}[B,C]+\dots} = \mathrm{e}^D\,, && (4.29)\\
D &= -a^2\partial_\nu A_\mu(x) + a^2\partial_\mu A_\nu(x) + a^2[A_\mu(x)\,,\, A_\nu(x)] + \mathrm{O}(a^3) && (4.30)\\
&= a^2\, F_{\mu\nu}(x) + \mathrm{O}(a^3) && (4.31)
\end{aligned}$$

We note that

$$D = D^a T^a\,, \quad \to \quad \mathrm{tr}\, D = 0\,. \tag{4.32}$$

This is true for the $\mathrm{O}(a^2)$ terms but also for the higher ones: they are formed from derivatives or from commutators. Commutators always can be written again as $C^a T^a$. Therefore we can write

$$\begin{aligned}
O_{\mu\nu}(x) &= \mathrm{tr}\,\mathrm{e}^D = N + \mathrm{tr}\, D + \tfrac{1}{2}\,\mathrm{tr}\, D^2 + \mathrm{O}(a^5) && (4.33)\\
&= N + \tfrac{1}{2}a^4\; \mathrm{tr}\,(F_{\mu\nu}(x))^2 + \mathrm{O}(a^5) && (4.34)\\
&= \mathrm{tr}\, U(p)\,,\; p = (x,\mu,\nu) && (4.35)
\end{aligned}$$



And finally have the Wilson plaquette action

$$S_{\mathrm{G}}[U] = \frac{1}{g_0^2}\sum_p \mathrm{tr}\,\{1 - U(p)\} = \frac{1}{g_0^2}\sum_x\sum_{\mu,\nu}\; \mathrm{tr}\, P(x,\mu,\nu)\,, \tag{4.36}$$

$$P(x,\mu,\nu) = 1 - U(x,\mu)\, U(x+a\hat{\mu},\nu)\, U(x+a\hat{\nu},\mu)^{-1}\, U(x,\nu)^{-1}\,. \tag{4.37}$$

Of course other forms are possible. In fact, take any small loop on the lattice (**a graph**), sum over all orientations. The result will be

$$\sum_{x,\mu,\nu} \tilde{O}_{\mu\nu}(x) = c_1 + c_2 a^4 \sum_{x,\mu,\nu}\; \mathrm{tr}\,(F_{\mu\nu}(x))^2 + \mathrm{O}(a^2)\,, \tag{4.38}$$

because there is no other gauge invariant field of dimension ≤ 4. It was therefore not really necessary to do the above calculation. By the same logics, there is no dimension five axis-permutation invariant field. This is why the next term is dimension 6, giving $\mathrm{O}(a^2)$ for the lattice spacing corrections.

4.2 Group integration

We remain in the pure gauge theory. In order to fully define the path integral we have to specify the integration measure. The variables are in the group $SU(3)$, so we want to know

$$\mathrm{d}U = ?\,, \quad U \in SU(N)\,. \tag{4.39}$$

The basic principle is gauge invariance, so we want the measure to be gauge invariant.

Parametrize the SU(N) matrices by

$$W(\omega) \;=\; \exp(\omega^a T^a)\,. \tag{4.40}$$

A gauge transformation gives

$$U(x,\mu) = W(\omega) \;\to\; U'(x,\mu) = \Lambda(x)\,U(x,\mu)\,\Lambda(x+a\hat{\mu})^{-1} = W(\omega') \tag{4.41}$$

$$W(\omega) \;\to\; W(\omega'(\omega))\,, \tag{4.42}$$

$$\text{we want } \mathrm{d}U'(x,\mu) \;=\; \mathrm{d}U(x,\mu)\,. \tag{4.43}$$

A naive expectation would be that $\mathrm{d}U \propto \prod_{a=1}^{N^2-1} \mathrm{d}\omega^a$, but it is not quite correct because SU(N) is a curved manifold. Eg. for SU(2) a parametrization is the following.

$$W = w^0 + i w^k \tau^k\,,\; w^\mu \in \mathbb{R}\,,\; w^\mu w^\mu = 1\,. \tag{4.44}$$

So the manifold is a 3-sphere. It has a curvature. We may choose (these ω are not the ω of $\exp(\omega^a T^a)$)

$$W = \sqrt{1-\omega^a\omega^a} + i\omega^a\tau^a\,,\; \omega^a \in [0,1]\,. \tag{4.45}$$

To account for the curvature we define a metric tensor on the manifold

$$G_{ab} = \;\mathrm{tr}\,(\frac{\partial W}{\partial \omega^a}\frac{\partial W^\dagger}{\partial \omega^b}) = G_{ba}\,, \quad G \geq 0\,, \quad \leftarrow \text{exercise} \tag{4.46}$$

In a transformation $\omega \to \omega'$ it changes to

$$G_{ab}(\omega') = \;\mathrm{Tr}\,(\frac{\partial W}{\partial \omega'^a}\frac{\partial W^\dagger}{\partial \omega'^b}) \;=\; \mathrm{Tr}\,(\frac{\partial W}{\partial \omega^c}\frac{\partial W^\dagger}{\partial \omega^d})\,\frac{\partial \omega^c}{\partial \omega'^a}\frac{\partial \omega^d}{\partial \omega'^b} = G_{cd}(\omega)\frac{\partial \omega^c}{\partial \omega'^a}\frac{\partial \omega^d}{\partial \omega'^b}\,. \tag{4.47}$$

As a remark (we do not need it): an invariant line element is

$$\mathrm{d}s^2 = G_{ab}\mathrm{d}\omega^a\mathrm{d}\omega^b = G_{ab}(\omega)\;\frac{\partial \omega^a}{\partial \omega'^c}\mathrm{d}\omega'^c\;\frac{\partial \omega^b}{\partial \omega'^d}\mathrm{d}\omega'^d = G_{cd}(\omega')\mathrm{d}\omega'^c\mathrm{d}\omega'^d\,. \tag{4.48}$$

Now take

$$\mathrm{d}W(\omega) = \sqrt{\det(G)}\prod_{a=1}^{N^2-1}\mathrm{d}\omega^a\,. \tag{4.49}$$

The pieces transform as

$$\det(G') \;\overset{\text{eq. (4.47)}}{=}\; \det(G)(\det(\frac{\partial \omega^a}{\partial \omega'^b}))^2 \tag{4.50}$$

$$\prod_{a=1}^{N^2-1}\mathrm{d}\omega'^a \;=\; \det(\frac{\partial \omega'^a}{\partial \omega^b})\prod_{c=1}^{N^2-1}\mathrm{d}\omega^c \tag{4.51}$$

$$\to\; \mathrm{d}W(\omega') = \mathrm{d}W(\omega)\,. \tag{4.52}$$

So we choose

$$\mathrm{d}U(x,\mu) = C\prod_{a=1}^{N^2-1}\mathrm{d}\omega^a\sqrt{\det G} \tag{4.53}$$

$$\int \mathrm{d}U = 1 \qquad \text{fixes } C\,. \tag{4.54}$$

This is the Haar measure, the invariant measure on the group.

Exercise

Show that for our SU(2)-parametrization

$$G_{ab} = \frac{\omega^a\omega^b + (1-\omega^c\omega^c)\,\delta_{ab}}{(1-\omega^c\omega^c)}\,, \tag{4.55}$$

$$\mathrm{d}U(x,\mu) = \mathit{const.}\times(1-\omega^c\omega^c)^{-1/2}\prod_a \mathrm{d}\omega^a = \frac{1}{\pi^2}\delta(1-w^\mu w^\mu)\mathrm{d}^4w\,. \tag{4.56}$$

So the parameter space of the group is a 3-sphere.

4.2.1 Some group integrals (for later)

What about

$$\langle U\rangle \equiv \int \mathrm{d}U\,U = \int \mathrm{d}(\Lambda^{-1}U)\,U = \int \mathrm{d}U'\,\Lambda U' = \Lambda\langle U\rangle \tag{4.57}$$

for all $\Lambda \in \mathrm{SU}(N)$. In particular for

$$\Lambda = \exp(i\,2\pi n/N) \equiv \exp(i\,2\pi n/N)\mathbb{1}\,, n = 0\ldots N-1\,, \quad \text{the center of SU}(N) \tag{4.58}$$

then

$$\langle U\rangle = \exp(i\,2\pi n/N)\langle U\rangle = \frac{1}{N}\sum_{n=0}^{N-1}\exp(i\,2\pi n/N)\langle U\rangle = 0 \tag{4.59}$$

$$\langle U^\dagger\rangle = 0\,. \tag{4.60}$$

A non-trivial group integral:

$$f_{ijkl} = \int \mathrm{d}U\,U_{ij}(U^\dagger)_{kl} = \int \mathrm{d}U\,U_{ij}(U_{lk})^* \tag{4.61}$$

$$= \int \mathrm{d}(\Lambda^{-1}U)\;U_{ij}(U^\dagger)_{kl} = \int \mathrm{d}(U\tilde\Lambda^{-1})\;U_{ij}(U^\dagger)_{kl} \tag{4.62}$$

$$= \int \mathrm{d}U\;(\Lambda U)_{ij}(U^\dagger\Lambda^\dagger)_{kl} = \int \mathrm{d}U\;(U\tilde\Lambda)_{ij}(\tilde\Lambda^{-1}U^\dagger)_{kl}. \tag{4.63}$$

For:

$$\text{fixed } j,k: \quad F_{il} = f_{ijkl}\,, \quad \Lambda F\Lambda^{-1} = F \quad \underbrace{\Rightarrow F = c\,\mathbb{1}}_{\text{exercise}} \tag{4.64}$$

$$\text{fixed } i,l: \quad G_{jk} = f_{ijkl}\,, \quad \tilde\Lambda G\tilde\Lambda^{-1} = G \quad \Rightarrow G = c'\,\mathbb{1} \tag{4.65}$$

$$\Rightarrow \quad f_{ijkl} = c\,\delta_{il}\delta_{jk} \tag{4.66}$$

It remains to determine c:

$$\sum_j f_{ijjl} = \int dU\, \delta_{il} = c \sum_j \delta_{il}\delta_{jj} \tag{4.67}$$

$$\rightarrow \quad c = 1/N\,. \tag{4.68}$$

Exercise

F is an $N \times N$ matrix. Prove that if $\Lambda F \Lambda^{-1} = F$ holds for all $\Lambda \in \mathrm{SU}(N)$ then $F = c\,\mathbb{1}$.
Hints: Start with $N = 2$. Find two special SU(2) matrices which allow to show $F = c\,\mathbb{1}$.
Embed SU(2) in SU(N) and use the $N = 2$ property to show it for all N.

4.3 Pure gauge theory

The path integral is now

$$Z = \int \mathrm{D}[U]\, \mathrm{e}^{-S_{\mathrm{G}}[U]}\,, \quad \mathrm{D}[U] = \prod_{x,\mu} dU(x,\mu)\,, \tag{4.69}$$

$$\langle O[U] \rangle = \frac{1}{Z}\int \mathrm{D}[U]\, O[U]\, \mathrm{e}^{-S_{\mathrm{G}}[U]} \tag{4.70}$$

$$U(x + L_\nu \hat{\nu}, \mu) = U(x,\mu)\,, \quad \text{PBC} \tag{4.71}$$

$$\Lambda(x + L_\nu \hat{\nu}) = \Lambda(x)\,, \quad \text{for the gauge transformations} \tag{4.72}$$

where $S[U] = S_{\mathrm{G}}[U]$ when matter fields are neglected. This is the pure gauge theory. We will always first work with the finite volume theory and then discuss the limit of large volume.

4.3.1 Gauge invariance

Consider some observable, $O[U]$, any polynomial of the fields $U(x,\mu)$. Its expectation value is the same as the expectation value of any gauge transform of it:

$$\langle O[U] \rangle = \frac{1}{Z}\int \prod_{x,\mu} dU(x,\mu)\, O[U]\, \mathrm{e}^{-S[U]} \tag{4.73}$$

$$= \frac{1}{Z}\int \prod_{x,\mu} dU^{\Lambda^{-1}}(x,\mu)\, O[U]\, \mathrm{e}^{-S[U]} \tag{4.74}$$

$$= \frac{1}{Z}\int \prod_{x,\mu} dU(x,\mu)\, O[U^\Lambda]\, \mathrm{e}^{-S[U]} = \langle O[U^\Lambda] \rangle\,. \tag{4.75}$$

We may define a projector onto the gauge invariant part of $O[U]$:

$$\mathbb{P}_0 O[U] = \int \prod_x d\Lambda(x)\, O[U^\Lambda]\,. \tag{4.76}$$

Then integrating above over $\Lambda(x)$ we see that

$$\langle \mathbb{P}_0 O[U] \rangle = \langle O[U] \rangle\,, \tag{4.77}$$

so one needs to consider only the gauge invariant part of any observable. Consider in particular an observable (e.g. a parallel transporter from y to x) which transforms as

$$O[U^\Lambda] = \Lambda(x) O[U] \Lambda^{-1}(y) \quad \to \quad \langle O[U] \rangle = 0\,. \tag{4.78}$$

Open parallel transporters (in contrast to closed loops) have no gauge invariant part. Their expectation values vanish. Such a local symmetry can't break spontaneously (Elitzur theorem).

4.3.2 Transfer Matrix

Hilbert space

Wave functionals of the spatial link variables $V(\mathbf{x}, k)$ with scalar product

$$\langle \psi' | \psi \rangle = \int \mathrm{D}[V] (\psi'[V])^* \, \psi[V]\,, \quad \mathrm{D}[V] = \prod_{\mathbf{x},k} \mathrm{d}V(\mathbf{x}, k)\,. \tag{4.79}$$

We will see that the physical Hilbert space consists of the gauge invariant wave functionals only:

$$\psi_{\mathrm{phys}}[V^\Lambda] = \psi_{\mathrm{phys}}[V]\,, \quad \Lambda(\mathbf{x}) \in \mathrm{SU}(3) \tag{4.80}$$

a projector onto gauge invariant wave functionals is

$$\hat{\mathbb{P}}_0 \, \psi[V] \;=\; \int \prod_{\mathbf{x}} \mathrm{d}\Lambda(\mathbf{x}) \, \psi[V^\Lambda]\,, \tag{4.81}$$

$$\hat{\mathbb{P}}_0 \, \psi_{\mathrm{phys}}[V] \;=\; \psi_{\mathrm{phys}}[V]\,. \tag{4.82}$$

Transfer matrix

Let us first remember quantum mechanics: the path integral with a finite time-spacing a is

$$Z \;=\; \mathrm{Tr}\, \hat{\mathbb{T}}^N\,, \quad N = T/a \tag{4.83}$$

$$\hat{\mathbb{T}} \, \psi(q) \;=\; \int \mathrm{d}q' \, \mathrm{e}^{-\Delta S(q,q')} \, \psi(q') \tag{4.84}$$

$$\Delta S(q, q') \;=\; \tfrac{1}{2} a V(q) + S_k(q, q') + \tfrac{1}{2} a V(q')\,, \quad S_k(q, q') = \frac{1}{2m} (\frac{q - q'}{a})^2\,. \tag{4.85}$$

So this is an integral operator on wave functions $\psi(q)$. We can write the pure gauge theory path integral in the same way

$$\hat{\mathbb{T}}\,\psi[V] = \int \mathrm{D}V'\,K[V,V']\,\psi[V'] \tag{4.86}$$

$$K[V,V'] = \int \prod_{\mathbf{x}} \mathrm{d}W(\mathbf{x})\mathrm{e}^{-\Delta S[V,W,V']}\,, \tag{4.87}$$

$$\Delta S[V,W,V'] = \tfrac{1}{2}S_p[V] + S_k[V,W,V'] + \tfrac{1}{2}S_p[V']\,, \tag{4.88}$$

$$S_p[V] = \frac{1}{g_0^2}\sum_{\mathbf{x}}\sum_{k,l=1}^{3} \operatorname{tr} P(\mathbf{x},k,l)\,. \tag{4.89}$$

$$S_k[V,W,V'] = \frac{1}{g_0^2}\sum_{\mathbf{x}}\sum_{k=1}^{3} \operatorname{tr}\left[P(\mathbf{x},k,0) + P(\mathbf{x},k,0)^\dagger\right]\,, \tag{4.90}$$

$$P(\mathbf{x},k,0) = 1 - V'(\mathbf{x},k)\,W(\mathbf{x}+a\hat{k})\,V(\mathbf{x},k)^{-1}\,W(\mathbf{x})^{-1}\,. \tag{4.91}$$

The kernel is real and symmetric, so the transfer matrix is hermitean. It is also bounded. Therefore the spectrum is discrete. (General theorem in functional analysis). Now perform a gauge transformation $\Lambda(\mathbf{x})$ just on the layer V

$$\Delta S[V,W,V'] = \Delta S[V^\Lambda, W^\Lambda, V']\,, \tag{4.92}$$

$$K[V,V'] = \int \prod_{\mathbf{x}} \mathrm{d}W(\mathbf{x})\mathrm{e}^{-\Delta S[V^\Lambda, W\Lambda^{-1}, V']}\,, \quad \text{choose } \Lambda = W \tag{4.93}$$

$$= \int \prod_{\mathbf{x}} \mathrm{d}W(\mathbf{x})\mathrm{e}^{-\Delta S[V^W,1,V']} \tag{4.94}$$

$$\rightarrow \quad \hat{\mathbb{T}} = \hat{\mathbb{P}}_0\hat{\mathbb{T}}_0 \tag{4.95}$$

$$K_0[V,V'] = \mathrm{e}^{-\Delta S[V,1,V']}\,, \tag{4.96}$$

and finally (perform a gauge trafo on both layers):

$$\hat{\mathbb{T}} = \hat{\mathbb{P}}_0\hat{\mathbb{T}}_0\hat{\mathbb{P}}_0\,. \tag{4.97}$$

We have learnt that the timelike gauge links in the path integral do not really corrspond to QM degrees of freedom, but represent a projector onto the gauge invariant subspace of the Hilbert space. The physical transfer matrix is defined in that space.

Physical states are gauge invariant!

Positivity

$$\hat{\mathbb{T}} > 0 \tag{4.98}$$

can be shown rigorously; ≥ 0 is a rather simple exercise. The rest requires some mathematics...