Introduction to QCD on the lattice

Rainer Sommer

DESY, Zeuthen

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  ... or open the Monvay/Münster

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$\longrightarrow$ numbers for phenomenology
- the connection between perturbative and non-perturbative phenomena

A rigorous definition of QCD

Wilson regularization

- strict positivity [Lüscher;Creutz]
  renormalizability (≡ contin. limit) shown to all order of PT [Reisz]
  all (flavor) vector symmetries

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- Conceptually clean framework if there ever is a doubt about an *n*-loop computation

A computational tool from "first principles"

- first principles

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2g_0^2}\,\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\} + \sum_f \overline{\psi}_f\{D + m_f\}\psi_f$$

$$\overbrace{\begin{bmatrix} F_\pi \\ m_\pi \\ m_{\rm K} \\ m_{\rm D} \\ m_{\rm B} \end{bmatrix}}^{\text{Experiment}} \quad \overset{\mathcal{L}_{\rm QCD}(g_0, m_f)}{\Longrightarrow} \quad \overbrace{\begin{bmatrix} \Lambda_{\rm QCD} \\ \hat{M} = (M_{\rm u} + M_{\rm d})/2 \\ M_{\rm s} \\ M_{\rm c} \\ M_{\rm b} \end{bmatrix}}^{\text{QCD parameters (RGI)}} \quad + \quad \overbrace{\begin{bmatrix} \xi \\ F_B \\ B_B \\ \dots \end{bmatrix}}^{\textbf{Predictions}}$$

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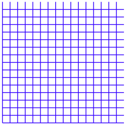
Discretization of $\mathcal{L}_{\rm QCD}$
with

- gauge invariance
- locality
- unitarity



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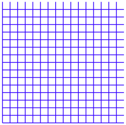

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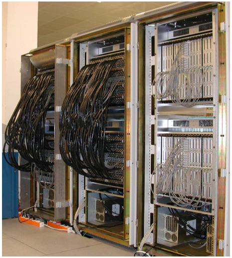

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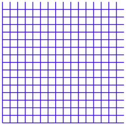

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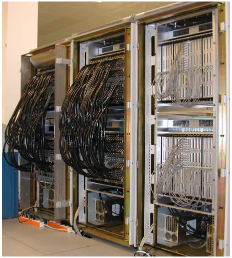

renormalization $\Downarrow$ continuum limit

low energy matrix elements

$$\pm \mathrm{O}\left(\frac{1}{\sqrt{\text{computer time}}}\right)$$

Basic steps of a computation

this was a bit simplified ... we need the basic steps

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- Formulate the problem in the Euclidean
- Discretize $\equiv$ Regularize and simulate
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- Continumm limit (remove the Regularization)

Formulation

- Easy (for the formulation!) case:

  Spectrum and simple matrix elements

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  an example: flavour currents:

$$A_\mu^{ij} = \overline{\psi}_i \gamma_\mu \gamma_5 \psi_j \,, \qquad ip_\mu F_{\rm K} = \langle K(p)| A_\mu^{\rm us}(0)|0\rangle$$

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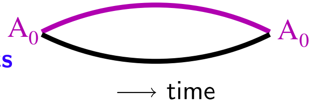

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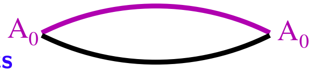

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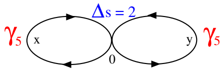

$$\frac{\int \mathrm{d}^3\mathbf{x} \mathrm{d}^3\mathbf{y} \, \langle A_0^{\rm su}(x)\, \mathcal{O}_{\Delta \rm s=2}(0) A_0^{\rm su}(y) \rangle}{\mathrm{d}^3\mathbf{x} \, \langle A_0^{\rm us}(x)\, A_0^{\rm su}(0) \rangle} \;\propto\; B_{\rm K}$$

Formulation

- easy case: Equilibrium thermodynamics
  finite temperature
  Euclidean $L_0 \times L^3$ spacetime,

$$\begin{aligned} \psi(x+L_0\hat{0}) &= -\psi(x)\,, \qquad \overline{\psi}(x+L_0\hat{0}) = -\overline{\psi}(x) \\ A_\mu^a(x+L_0\hat{0}) &= +A_\mu^a(x) \end{aligned}$$

then

$$\text{path integral} = \mathrm{Tr}\, e^{-\mathbb{H}L_0} = \mathrm{Tr}\, e^{-\mathbb{H}/T}$$

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and finite chemical potential μ_q

$$\mathcal{L}_{\rm QCD} \rightarrow \mathcal{L}_{\rm QCD} + \mu_q \underbrace{\overline{\psi}\gamma_0\psi}_{\text{number density}}$$

$$\text{path integral} = \mathrm{Tr} e^{-(\mathbb{H}-\mu_q N_q)/T}$$

$\longrightarrow$ thermodynamic properties, eg. EOS talk by S. Katz

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  talks by Giusti, Lubicz
- Impossible (?) cases
  H H' → 5 H
  The mass of a di-quark
  The gluon condensate
  ...

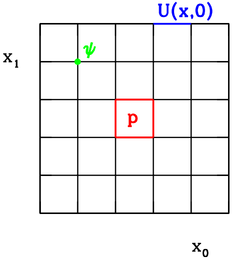
Discretize and "simulate"

lattice: $x_\mu = an_\mu,\ n_\mu \in \mathbb{Z}$

quarks: $\psi(x)$ on lattice points

gluons: $U(x,\mu) = \mathcal{P}\exp\left\{a\int_0^1 \mathrm{d}t A_\mu(x+a(1-t)\hat\mu)\right\}$ $\in \mathrm{SU}(3)$ on links

Euclidean action: $S = S_\mathrm{G} + S_\mathrm{F}$



$$S_\mathrm{G} = \frac{1}{g_0^2}\sum_p \mathrm{tr}\,\{1 - U(p)\}\,,$$

$$S_\mathrm{F} = a^4\sum_x \overline{\psi}(x)(D(U)+m)\psi(x)$$

$D(U)$: discretized Dirac operator

Observables:

$$\langle O\rangle = \frac{1}{Z}\int \mathrm{D}[\psi]\mathrm{D}[\bar\psi]\mathrm{D}[U]\ O\ \mathrm{e}^{-S}$$

$$Z : \langle 1\rangle = 1$$

Can be interpreted as a statistical system in $3+1=4$ dimensions.
$(S \to H/(kT)\,,\ \ 1/(kT) \propto 1/g_0^2)$

First principles evaluation of $\langle O\rangle$ by MC method

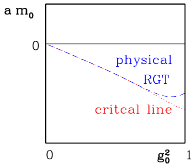
statistical errors $\propto 1/\sqrt{\text{computer time}}$.

Renormalization and continuum limit: remove the regularization

For simplicity discussion for $N_\mathrm{f} = 2$ mass-degenerate quarks

- In QCD: critical line with (almost) massless quarks, massless π's;
  Continuum limit: $(am_\mathrm{hadron} \to 0)$ at $g_0 \to 0$

  Along RGT $\equiv$ fixed physical quark mass:



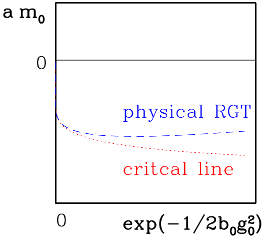
$$a\, m_\mathrm{hadron} \sim \mathrm{C}_\mathrm{hadron}\, \mathrm{e}^{-1/(2b_0 g_0^2)} (2b_0 g_0^2)^{-b_1/2b_0^2} \times \underbrace{[1 + \mathrm{O}(g_0^2)]}_{\text{badly convergent}}$$

$$b_0 = (11 - \tfrac{2}{3} N_\mathrm{f})/(4\pi)^2$$

- More precisely

  fix $m_\pi^2 / m_{\rm nucleon}^2 = 137/938 \leftarrow$ experiment
  fix $m_{\rm nucleon} = 938\,{\rm MeV} \leftarrow$ experiment

  $$\Rightarrow \quad a = \frac{a\, m_{\rm nucleon}}{938\,{\rm MeV}} \qquad \leftarrow \text{“setting the scale”}$$



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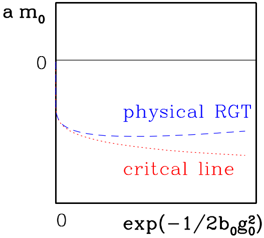

other hadron masses and e.g. F_π are predictions up to ${\rm O}(a)$ or ${\rm O}(a^2)$ errors

bare parameters $\longrightarrow$ renormalized ones $\longleftarrow$ hadronic renormalization scheme
$g_0\,, m_0$ $\quad m_\pi\,, m_{\rm nucleon}$

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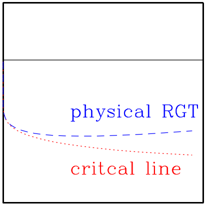

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$$\left.\frac{m_{\rm hadron}}{m_{\rm nucleon}}\right|_{\rm lattice} = \left.\frac{m_{\rm hadron}}{m_{\rm nucleon}}\right|_{\rm QCD} \times [1 + \mathrm{O}(a^n)]$$

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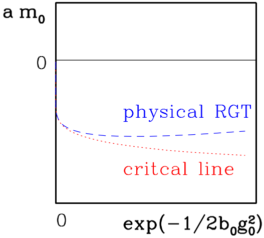

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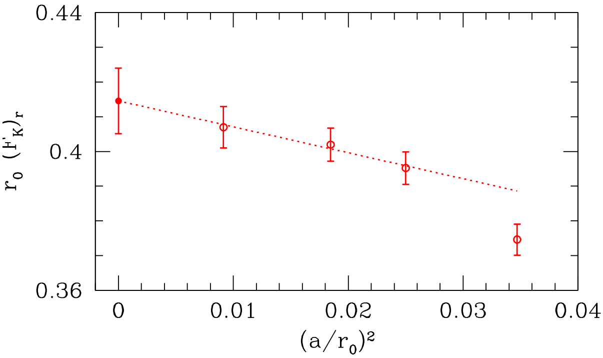
- better: see

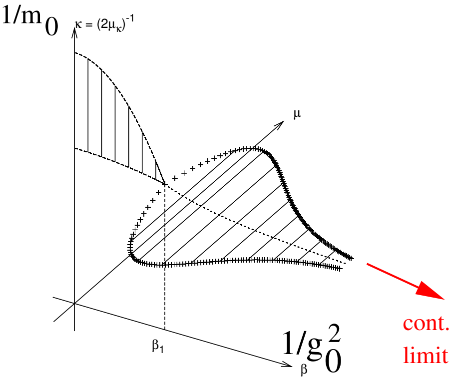


expl. from [Garden et al. 1999] quenched

Continuum limit continued

- beware
  of phase transitions

  example from tmQCD
  [Sharpe et al.; Münster; Scorzato]
  [Farchioni et al., χLF Coll.]



The challenges for lattice QCD as a computational tool

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furthermore:

- short physical distances to connect to PT → Knechtli
- complex actions (finite μ_q) very difficult for MC-technique → Katz
  ($S_{\rm eff}(U)$ is not real for finite μ_q)

The work horse of LQCD: the HMC [Duane, Kennedy, Pendleton, Roweth]

$$\begin{aligned} \langle O(\Phi)\rangle &= \frac{1}{Z}\int_\Phi O\mathrm{e}^{-S(\Phi)} \qquad \Phi=\{U(x,\mu),\psi(x),\overline{\psi}(x)\} \\ &= \frac{1}{N}\sum_{i=1}^{N} O(\Phi_i) \quad \text{for} \quad P(\Phi)\propto \mathrm{e}^{-S(\Phi)}>0 \end{aligned}$$

local bosonic actions: Φ_i with Metropolis, heatbath, OR algo's: fast

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QCD: integrate out fermions (Grassmann)

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rewrite with bosonic field ϕ

$$\langle O(U)\rangle = \frac{1}{Z}\int_{U,\phi,\phi^\dagger}[O(U)]_{\mathrm{F}}\,\exp(-S_{\mathrm{G}}(U)-\frac{N_{\mathrm{f}}}{2}\,\phi^\dagger[(D(U)+m_0)^\dagger(D(U)+m_0)]^{-1}\phi)$$
$$\equiv \frac{1}{Z}\int_{U,\phi,\phi^\dagger}[O(U)]_{\mathrm{F}}\mathrm{e}^{-S_{\mathrm{PF}}}$$

introduce a (classical) hamiltonian with U as coordinates and S_{PF} as potential and conjugate variables to U (momenta p)

$$H = \sum_{x,\mu} \tfrac{1}{2} p^2(x,\mu) + S_{\mathrm{PF}}(U,\phi)$$

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algorithm

- U
- p, ϕ with $P(\phi) \propto \mathrm{e}^{-S_{\rm PF}(U,\phi)}$, $P(p) \propto \mathrm{e}^{-\frac{1}{2}p^2}$
- Hamiltonian evolution

  (of course discretised with step-size δt)

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}H}{\mathrm{d}p} = p$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{-\mathrm{d}H}{\mathrm{d}U} = -\frac{-\mathrm{d}S_{\rm PF}}{\mathrm{d}U}$$

- U'; metropolis accept/reject (corrects step size errors) introduce a (classical) hamiltonian with U as coordinates and $S_{\rm PF}$ as potential and conjugate variables to U (momenta p)

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$\longrightarrow P(U) \propto \mathrm{e}^{-S_{\rm eff}(U)}$

- the expensive step

$$\frac{-\mathrm{d}S_{\mathrm{PF}}}{\mathrm{d}U} = \frac{N_\mathrm{f}}{2}\,\phi^\dagger[(D(U)+m_0)^\dagger(D(U)+m_0)]^{-1}\frac{\mathrm{d}D}{\mathrm{d}U}(D(U)+m_0)^{-1}\phi)$$

- the expensive step

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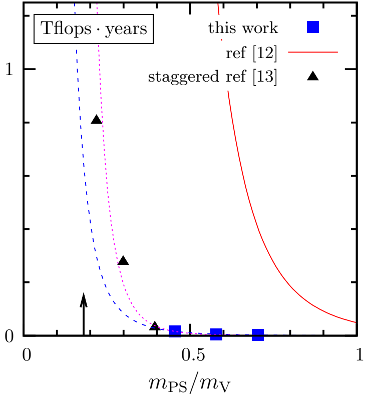
- in general: small quark masses are very difficult to reach
- reach up+down quark masses from heavier ones

guided by ChPT $\rightarrow$ later

The costs: e.g. for $a = 0.08\,\mathrm{fm}$, 1000 configurations U_i

($0.05\,\mathrm{fm}$ a factor ∼30)

- Berlin "wall" ——
  Lattice 2001, Berlin



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  ("Hasenbusch trick")
  [Hasenbusch; Jansen & Hasenbusch; ...]

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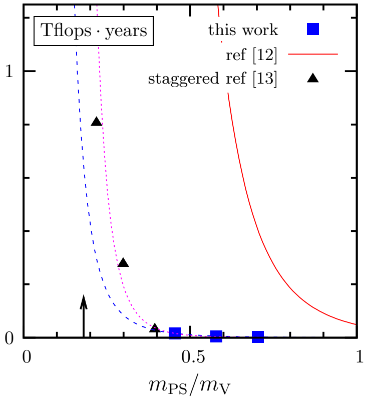
  [Urbach et al., 2005]



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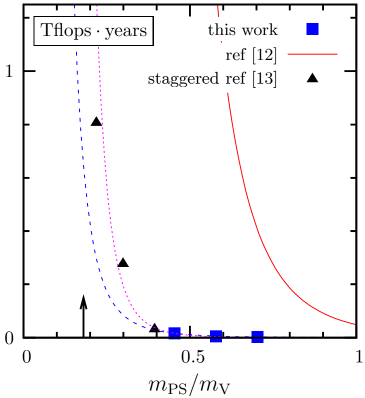
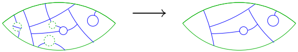


- more radical: domain decomposition [Lüscher] ⟶ Giusti
  profiting from locality of QCD

The quenched approximation

$$\langle O(U)\rangle = \frac{1}{Z}\int_{\mathrm{U}} \underbrace{[O(U)]_{\mathrm{F}}}_{[D(U)+m_0]^{-1}} \exp(-S_{\mathrm{G}}(U) + N_{\mathrm{f}} \underbrace{\mathrm{Tr}\,\ln(D(U)+m_0)}_{\text{non}-\text{local drop it}})$$

$N_{\mathrm{f}} \longrightarrow 0$ limit, no sea quarks



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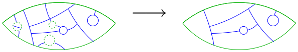
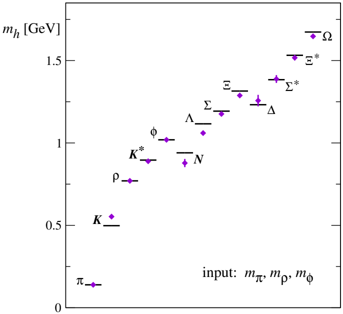


- quenched hadron mass spectrum cont. limt [CP-PACS,2002]



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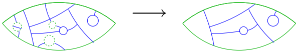
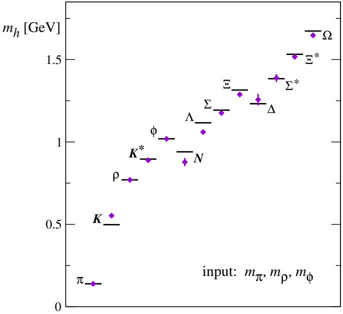




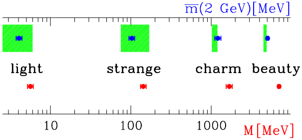
- quenched
  hadron mass spectrum
  cont. limt
  [CP-PACS,2002]

- Amazing success, but qQCD is not QCD.
  Factor > 100 work for the remaining 10-x%

typically 90+% of the answer
no better way?

Heavy quarks: The particular challenge for B-physics

multiple scale problem
always difficult
for a numerical treatment



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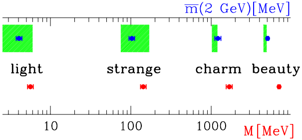
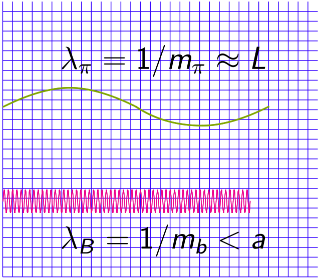


- Take a large lattice as it is possible in the quenched approximation



light quarks are too light
$\longrightarrow$ treat by an extrapolation

b-quark is too heavy

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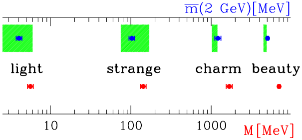

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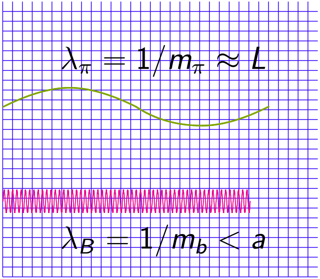


light quarks are too light
→ treat by an extrapolation

b-quark is too heavy

- Need an effective theory for the b-quark: HQET, NRQCD

[E. Eichten, 1988; E. Eichten & B. Hill 1990; Caswell & Lepage; Lepage & Thacker]

- b-quark is too heavy:

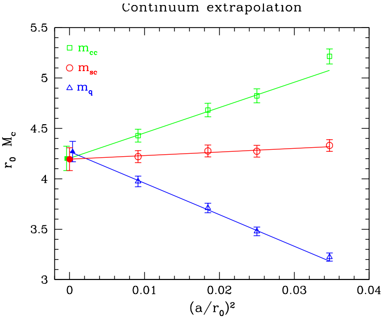
  charm: [Sint & Rolf]

  3 definitions of charm quark mass differ by *a*-effects

  charm just doable

  $aM_\mathrm{b} \approx 4aM_\mathrm{c}$!

  a possible alternative (under investigation): Symanzik improvement to all orders in *am* [Aoki,Kuramashi]
  ∼ Fermilab action [El Khadra,Kronfeld,Mackenzie]



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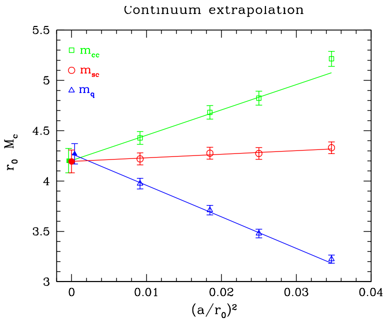

- for b-quarks $a \longrightarrow 0$ can't be controlled in this way

Renormalization (with mixing)

1. Bare parameters of QCD, $g_0, m_{0,\mathrm{i}}$

   $\longrightarrow$ fundamental parameters, Λ, $M_{0,\mathrm{i}}$ (RGI's).

   prediction of Λ, $M_{0,\mathrm{i}}$ with hadron spectrum as input. $\longrightarrow$ Knechtli
2. Renormalization of composite operators

$$\text{e.g.} \qquad \langle \overline{K_0} | \mathbb{H}_{\mathrm{weak}} | K_0 \rangle_{\mathrm{SM}} \qquad \leftrightarrow \qquad C(\mu) \langle \overline{K_0} | \mathcal{O}_{\Delta \mathrm{s}=2}(\mu) | K_0 \rangle_{\mathrm{QCD}}$$

$$\uparrow$$

matching

similar

$$\langle B | A_0^{\mathrm{bu}} | 0 \rangle \leftrightarrow C(\mu) \langle B | A_0^{\mathrm{bu,stat}}(\mu) | 0 \rangle + \mathrm{O}(1/m_b)\,, \qquad \mu = m_b$$

- matching usually done in the continuum, dimensional regularization, $\overline{\mathrm{MS}}$-renormal. of coupling and masses
  still have to renormalize the bare lattice operators and connect to $\overline{\mathrm{MS}}$

Renormalization (with mixing)

3. Weak currents

$$(A_\mathrm{R})_\mu = Z_\mathrm{A}(g_0)A_\mu$$

- special case of composite operators: renormalization is scale independent
- can be fixed by χ WI's [Bochicchio et al.]
- $Z_\mathrm{A} = 1$ if the action has exact (lattice) χ symmetry

4. Coefficients in effective (lattice) actions

- e.g. lattice HQET

5. Others, in particular hard problems like

$$\mathcal{O}_\mathrm{R} = Z(\mu)\left\{\mathcal{O} + \frac{c_1(g_0)}{a}\mathcal{O}_1 + \frac{c_2(g_0)}{a^2}\mathcal{O}_2 + \frac{c_3(g_0)}{a^3}\mathcal{O}_3\right\}$$

- e.g. $\Delta\mathrm{I} = 1/2$ problem

Renormalization (with mixing)

1.A Take a physical observable, expandable in renormalized perturbation theory, e.g.

$$F(r) = \frac{4}{3}\frac{1}{r^2}\left\{\alpha_{\overline{\rm MS}}(\mu) + c_1[\alpha_{\overline{\rm MS}}(\mu)]^2 + \ldots\right\}, \quad \mu = 1/r$$

$$\equiv \frac{4}{3}\frac{1}{r^2}\,\alpha_{\rm qq}(\mu) \qquad \text{physical coupling}$$

compute the physical coupling for large μ; $\rightarrow \Lambda$
in practice: the Schrödinger functional $\rightarrow$ Knechtli

1.B Use bare (UV) quantity, expandable in bare (lattice) perturbation theory

e.g. $\quad P = \frac{1}{N}\langle \mathrm{tr}\, U(p)\rangle \qquad \alpha_\Box \equiv -\frac{1}{C_{\rm F}\pi}\ln(\underbrace{P}_{\text{from MC!}})$

then $\quad \alpha_{\overline{\rm MS}}(s_0 a^{-1}) = \alpha_\Box + 0.614\alpha_\Box^3 + {\rm O}(\alpha_0^4) + {\rm O}(a)$

or (see later) α_V

properties

A) "physical" observable

- $+$ only continuum PT
- $+$ separates continuum limit (a^n) and renormalization effects (α^m)
- $-$ needs "window"

$$L^{-1} \ll \Lambda \ll \mu \ll a^{-1}$$

- $+$ modification $L^{-1} \equiv \mu$ (use the finite volume); then $L \gg a$ is sufficient;
  large μ by recursion $\mu_0 \to 2\mu_0 \to \ldots \to 2^n\mu_0$

B) UV (cutoff) quantity

- $+$ easy, quick (all you need is a hadron mass)
- $-$ mixing of discretization errors (a^n) and renormalization effects (α^m): $\alpha_{\overline{\rm MS}}(s_0 a^{-1})$
- $-$ difficult (for $N_{\rm f} > 0$ impossible) to reach high μ

2. Renormalization of composite operators

Example

$$\mathcal{O}(x) \equiv \mathcal{O}_{\Delta \mathrm{s}=2}(x) = \bar{s}(x)\gamma_\mu^{\mathrm{L}} d(x)\, \bar{s}(x)\gamma_\mu^{\mathrm{L}} d(x)$$

assume exact χ symmetry $\rightarrow$ $\mathcal{O}_{\mathrm{R}}(x) = Z_{\mathcal{O}}(\mu)\mathcal{O}(x)$

a possible renormalization condition, intermediate scheme

$(P(x) = \bar{d}(x)\gamma_5 s(x),\ P^+(x) = \bar{s}(x)\gamma_5 d(x))$

$$\begin{aligned} \langle P_{\mathrm{R}}(x_1)\mathcal{O}_{\mathrm{R}}(0)P_{\mathrm{R}}(x_2)\rangle &= Z_{\mathrm{P}}^2(\mu)Z_{\mathcal{O}}(\mu)\langle P(x_1)\mathcal{O}(0)P(x_2)\rangle \qquad \text{e.g. } x_1 = -x_2 \\ &= \langle P(x_1)\mathcal{O}(0)P(x_2)\rangle_{\text{tree level}} \qquad x_1^2 = x_2^2 = \mu^{-2} = x^2 \\ \langle P_{\mathrm{R}}(x)P_{\mathrm{R}}^+(0)\rangle &= Z_{\mathrm{P}}^2(\mu)\langle P(x)P(0)\rangle = \langle P(x)P(0)\rangle_{\text{tree level}} \end{aligned}$$

- ⋆ gauge invariant
- ⋆ for large μ connected to other schemes

$$\mathcal{O}_{\mathrm{R}}(\mu) = \mathcal{O}_{\overline{\mathrm{MS}}}(\mu) \times [1 + c_1\alpha_{\overline{\mathrm{MS}}} + \ldots]$$

- ⋆ problems of such a ren. cond.
  - – signals in MC
  - – need PT $(c_1 \ldots)$
  - – need window: $L^{-1} \ll \Lambda \ll \mu \ll a^{-1}$

$\longrightarrow$ not practical

Renormalization (with mixing)

alternative 1:

- ⋆ fix gauge
- ⋆ $P(x) = \bar{d}(x)\gamma_5 s(x) \rightarrow \bar{d}(y_1)s(y_2)$ "quark states"
- ⋆ go to momentum space → MOM-scheme ("RI-MOM")
- – on-shell improvement does not work
- – off-shell improvement with non-gaugeinvariant terms
- + rather simple, "universal"

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alternative 2:

- ⋆ renormalization in finite volume, in particular in the SF
  $L \gg a$ is sufficient;
  large μ by recursion
- + in the SF: use boundary quark fields instead of $P(x)$
- + gauge invariant
- + on-shell improvement does work
- – need to do PT ourselves (usually 1-loop, sometimes 2)

4. Coefficients in effective (lattice) actions

Coefficients in effective (lattice) actions e.g. lattice HQET

$$S_{\rm QCD}(\ldots, M) \quad \leftrightarrow \quad S_{\rm HQET}(\ldots, c_{\sigma\cdot\mathbf{B}}(g_0), c_{\mathbf{D}}^2(g_0), \ldots)$$

$$\uparrow$$

matching

Matching, in principle ($M = m_b$):

$$O_{\rm QCD}(\ldots, M) = O_{\rm HQET}(\ldots, c_{\sigma\cdot\mathbf{B}}(g_0), c_{\mathbf{D}}^2(g_0), \ldots) + {\rm O}((\tfrac{|\mathbf{p}|}{M})^n, (\tfrac{\Lambda}{M})^n) + {\rm O}(a^m)$$

This may be done non-perturbatively [Heitger, S.]
(again using a finite volume for the matching step)

Mixing of operators of different dimension

appears e.g. in effective theories such as HQET

$$\mathcal{O}_{\mathrm{R}} = Z(\mu)\left\{\mathcal{O} + \frac{c(g_0)}{a}\mathcal{O}_1 + \frac{d(g_0)}{a^2}\mathcal{O}_2\right\}$$

perturbation theory is not enough for c, d:

PT: $c = c_1\, g_0^2 + c_2 g_0^4 + \ldots + c_n g_0^{2n}$ (take $d = 0$)

- perturbative uncertainty in $\mathcal{O}_{\mathrm{R}}$:

$$\delta\mathcal{O}_{\mathrm{R}} \sim \frac{g_0^{2n+2}}{a} \sim \frac{1}{\{2b_0 \ln(1/a\Lambda)\}^{n+1}\, a} \to \infty$$

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  no continuum limit

- it is a more general QCD problem: clean computation of power corrections needs full non-perturbative treatment of the leading term

And it is possible: e.g. $1/m$ corrections in HQET

it is possible with full non-perturbative renormalization

experiment		Lattice with $am_{\rm q} \ll 1$
$m_{\rm B} = 5.4\,\mathrm{GeV}$	$L_2 = 2L_1$	$\Phi_1(L_1, M), \Phi_2(L_1, M)$
↓		↓
$\Phi_1^{\rm HQET}(L_2), \Phi_2^{\rm HQET}(L_2)$	← $\sigma_{\rm m}(u_1)$; $\sigma_1^{\rm kin}(u_1), \sigma_2^{\rm kin}(u_1)$	$\Phi_1^{\rm HQET}(L_1), \Phi_2^{\rm HQET}(L_1)$

[ALPHA Collaboration : Heitger & S; Heitger & Wennekers; Heitger, Jüttner,S & Wennekers]

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$\downarrow$		$\downarrow$
$\Phi_1^\mathrm{HQET}(L_2), \Phi_2^\mathrm{HQET}(L_2)$	$\xleftarrow[\sigma_1^\mathrm{kin}(u_1),\, \sigma_2^\mathrm{kin}(u_1)]{\sigma_\mathrm{m}(u_1)}$	$\Phi_1^\mathrm{HQET}(L_1), \Phi_2^\mathrm{HQET}(L_1)$

[ALPHA Collaboration : Heitger & S; Heitger & Wennekers; Heitger, Jüttner,S & Wennekers]

very recent (quenched!): m_b, $\overline{\mathrm{MS}}$-scheme, cont.lim.

$$
\begin{aligned}
m_\mathrm{b}^\mathrm{stat}(m_\mathrm{b}) &= 4350(64)\,\mathrm{MeV} & \leftarrow \pm\mathrm{O}(\Lambda^2/m_\mathrm{b}) \\
m_\mathrm{b}^\mathrm{stat}(m_\mathrm{b}) + m_\mathrm{b}^1(m_\mathrm{b}) &= 4301(70)\,\mathrm{MeV} & \leftarrow \pm\mathrm{O}(\Lambda^3/m_\mathrm{b}^2)
\end{aligned}
$$

[ALPHA Collaboration : Della Morte, Garron, Papinutto, S.]

Effective field theories for (lattice) QCD

$|p_\mu| \ll \mathcal{M}$

low energy, small momentum expansions
derivative expansions e.g.
$\mathcal{L}_{\rm HQET} = \overline{\psi}_{\rm h} D_0 \psi_{\rm h} + \frac{1}{m_{\rm b}} \overline{\psi}_{\rm h} D_k D_k \psi_{\rm h}$

- $\mathcal{M} = m_{\rm b}$ — HQET
- $\mathcal{M} = a^{-1}$ — Symanzik effective theory
- $\mathcal{M} = T =$ Temperature — dimensional reduction
- $\mathcal{M} = 4\pi F_\pi$ and $B \times m_{\rm quark} = {\rm O}(|p_\mu|^2)$, $B = -\langle\overline{\psi}\psi\rangle / F_\pi^2$ — ChPT

renormalizable order by order in the expansion → will yield
(finite number of counterterms) the asymptotic expansion in $|p_\mu|/\mathcal{M}$

very useful (even necessary?)

new developments: combination of ChPT and Symanzik effective theory:

WChPT, SChPT

Symanzik effective theory

At energies, momenta far below the cutoff $1/a$, lattice QCD is equivalent to a continuum theory with effective action (on shell)

$$
\begin{aligned}
S_{\mathrm{eff}} &= S_{\mathrm{QCD}} + aS_1 + a^2 S_2 + \ldots \\
S_1 &= \int \mathrm{d}^4x\; c(g_0)\,\overline{\psi}\,\sigma_{\mu\nu}F_{\mu\nu}\psi + \text{less relevant} \\
S_2 &= \int \mathrm{d}^4x\; c'(g_0)\;\mathrm{tr}\,D_\rho F_{\mu\nu} D_\rho F_{\mu\nu} + \ldots
\end{aligned}
$$

$\Rightarrow$ O(a) cutoff effects in on-shell matrix elements can be canceled by adding

$$
a^5 \sum_x \tfrac{i}{4} c_{\mathrm{sw}} \overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \qquad c_{\mathrm{sw}} = 1 + \mathrm{O}(g_0^2),
$$

to the lattice action [Symanzik, Lüscher & Weisz, Sheikholeslami & Wohlert,..., ALPHA Collaboration]

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$$S_{\rm eff} = S_{\rm QCD} + a S_1 + a^2 S_2 + \dots$$
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$$S_2 = \int \mathrm{d}^4 x \; c'(g_0)\; \mathrm{tr}\, D_\rho F_{\mu\nu} D_\rho F_{\mu\nu} + \dots$$

$\Rightarrow$ O(a) cutoff effects in on-shell matrix elements can be canceled by adding

$$a^5 \sum_x \tfrac{i}{4} c_{\rm sw} \overline{\psi} \sigma_{\mu\nu} F_{\mu\nu} \psi, \qquad c_{\rm sw} = 1 + \mathrm{O}(g_0^2),$$

to the lattice action [Symanzik, Lüscher & Weisz, Sheikholeslami & Wohlert,..., ALPHA Collaboration]

- $\overline{\psi}\, \sigma_{\mu\nu} F_{\mu\nu} \psi$ is not chirally symmetric (so is the Wilson action)

Symanzik effective theory

At energies, momenta far below the cutoff $1/a$, lattice QCD is equivalent to a continuum theory with effective action (on shell)

$$\begin{aligned} S_{\rm eff} &= S_{\rm QCD} + aS_1 + a^2 S_2 + \ldots \\ S_1 &= \int \mathrm{d}^4x\, c(g_0)\, \overline{\psi}\,\sigma_{\mu\nu} F_{\mu\nu}\psi + \text{less relevant} \\ S_2 &= \int \mathrm{d}^4x\, c'(g_0)\, \mathrm{tr}\, D_\rho F_{\mu\nu} D_\rho F_{\mu\nu} + \ldots \end{aligned}$$

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- $\overline{\psi}\,\sigma_{\mu\nu} F_{\mu\nu}\psi$ is not chirally symmetric (so is the Wilson action)
- other actions exist, which have enough chiral symmetry to enforce $c(g_0) = 0$ "automatic $\mathrm{O}(a)$ improvement"

The Zoo of fermion actions

- Wilson: $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$ broken $\rightarrow SU_V(N_f) \times U_V(1)$
  $\mathrm{O}(a)$-effects
  χ symmetry only in the continuum limit — keep $m_0 > \mathrm{O}(a\Lambda^2)$
  mixing of operators of different dimension **and of different chirality**
- $\mathrm{O}(a)$-improved Wilson: the same but $\mathrm{O}(a^2)$ effects $\longrightarrow$
  [Sheikholeslami & Wohlert; ...; ALPHA Collaboration]
- tmQCD: mass term (per doublet) $m\overline{\psi}\gamma_5\tau^3\psi$ [Frezzotti, Grassi, Sint & Weisz]
  simplified $\mathrm{O}(a)$ improvement
  solves many mixing problems
  breaks parity to P_F — breaks flavor symmetry
  at maximum twist: automatic $\mathrm{O}(a)$ improvement: [Frezzotti & G.C.Rossi]
- Domain Wall fermions [Kaplan; Furman & Shamir]
  good χ symmetry — expensive (factor ≈ 20)
- classically perfect [Hasenfratz & Niedermayer]
  good χ symmetry — expensive (factor $\approx 10 - 50$)
  small a-effects $\longrightarrow$
- Ginsparg-Wilson type [Laliena, Hasenfratz, Niedermayer; Neuberger]
  exact χ symmetry [Lüscher] — very expensive (factor $\approx 10 - 100$)
  automatic $\mathrm{O}(a)$ improvement
- staggered [Kogut&Susskind] 4 tastes, but often reduced to 1 by "4-th root trick"
  (vector) flavor symmetry broken (pions with different masses)
- highly improved; FLIC, ...

Chiral perturbation theory including a-effects

low energy effective Lagrangian for QCD: χ Lagrangian

[Weinberg; Gasser & Leutwyler]

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_2 + \mathcal{L}_4 + \dots \\
\mathcal{L}_2 &= \frac{F^2}{4}\,\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) + \frac{F^2}{2}\,\mathrm{Tr}\left(\chi^\dagger \Sigma + \chi \Sigma^\dagger\right) \\
\mathcal{L}_4 &= \dots + L_4\,\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\,\mathrm{Tr}\left(\Sigma^\dagger \chi + \chi^\dagger \Sigma\right) \\
&\quad + L_5\,\mathrm{Tr}\left[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \left(\Sigma^\dagger \chi + \chi^\dagger \Sigma\right)\right] \\
&\quad + L_6 \left[\,\mathrm{Tr}\left(\Sigma^\dagger \chi + \chi^\dagger \Sigma\right)\right]^2 + L_7 \left[\,\mathrm{Tr}\left(\Sigma^\dagger \chi - \chi^\dagger \Sigma\right)\right]^2 \\
&\quad + L_8\,\mathrm{Tr}\left(\chi^\dagger \Sigma \chi^\dagger \Sigma + \chi \Sigma^\dagger \chi \Sigma^\dagger\right) + \dots + L_{12}\,\mathrm{Tr}\chi\chi^\dagger
\end{aligned}
$$

with

$$
\mathcal{M} \equiv \mathrm{diag}(m_u, m_d, m_s) \quad \Sigma \equiv \exp\left(\frac{i2T_a\pi_a}{F}\right) \quad \chi \equiv 2\mathcal{M}B
$$

and a − terms: $a\,\overline{\psi}\,\sigma_{\mu\nu}F_{\mu\nu}\psi \rightarrow \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\,\mathrm{Tr}\left(\Sigma^\dagger \rho + \rho^\dagger \Sigma\right) + \dots$

$\rho = a\,c_\rho(g_0)\,\mathbf{1}$

[Sharpe & Singleton 1998; Sharpe & Shoresh 2002; Rupak & Shoresh 2002] and a − terms: $a\,\overline{\psi}\,\sigma_{\mu\nu}F_{\mu\nu}\psi \rightarrow \mathrm{tr}\,\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right)\,\mathrm{tr}\,\left(\Sigma^\dagger\rho + \rho^\dagger\Sigma\right) + \ldots$

$\rho = a\,c_\rho(g_0)\,\mathbf{1}$

Predicts, for $m_q \ll \Lambda_{QCD}$, $a \ll 1/\Lambda_{QCD}$, e.g. (quarks degenerate)

$$\frac{m_\pi^2}{\mu} = 1 + \underbrace{\gamma}_{\uparrow}\log\mu + \underbrace{\alpha}_{L_i} + \underbrace{\alpha'}_{L_i}\bar{\rho} + \underbrace{\omega}_{W_i}\bar{\rho} + \mathrm{O}(\mu^2, \mu\bar{\rho}, \bar{\rho}^2)$$

prediction low energy constants: from fits

$$\mu = Bm_\mathrm{q}, \quad \bar{\rho} = ac_\rho(g_0)$$

when m_q, a are small enough:

functional form for extrapolations $m_\mathrm{q} \rightarrow m_\mathrm{u}, m_\mathrm{d}$, $a \rightarrow 0$ is known

There are of course extensions to $\mathrm{O}(a^2)$ very active right now.

A recent revolution?

- some transparencies taken from C. Davies
  HEP2005 International Europhysics Conference on High Energy Physics

A recent revolution?

- ## Take-home message

  - There has been a revolution in lattice QCD since 2003
  - Quenched approximation (ignores sea quarks) is dead - stop quoting results from it
  - Lattice QCD now delivering fully unquenched results: hadron masses that agree with expt; precise parameters of QCD; matrix elements relevant to CKM physics.

A recent revolution?

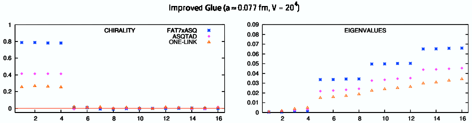
- Improved staggered quarks do not solve doubling problem

  4 'tastes' of quark instead of 1 - if all same, 'divide by 4' :

  $det(M) \rightarrow det^{1/4}(M)$ 

  'Taste-changing' interactions mess this up, but vanish as a^2

  NEW theoretical work encouraging, e.g. eigenvalues (M) divides into quartets, Index Theorem obeyed etc.



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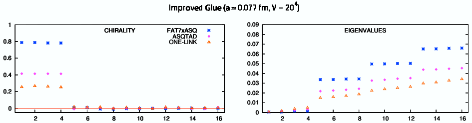
  

  Follana et al; Durr et al; Adams; Peardon et al; Shamir

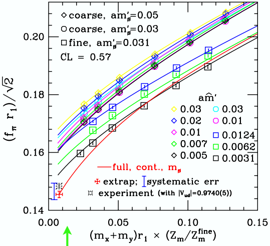
- No Lagrangian formulation known for this “fourth root trick”
  a discussion in the lattice community:

  Q: what are the quark fields?
  Q: is there a continuum limt? is it QCD?

  A: comparison to experiment   Q: a good model?

A recent revolution?

- ## Update on light decay constants



$$\pi(K \to \mu\pi) = \frac{G_F^2 m_K m_\mu^2}{8\pi}\left(1 - \frac{m_\mu^2}{m_K^2}\right) f_K^2 V_{us}^2$$

Use staggered chiral pert. th. to extrap. to physical pion mass (Aubin + Bernard) to take account of disc. errors. NEW 2005 - configs with second m_s.

Improved chiral extraps for f_π, f_K

$$\frac{f_K}{f_\pi} = 1.200(4)\left(^{+17}_{-5}\right)$$

Assuming V_{ud}

$$V_{us} = 0.2238\left(^{+12}_{-32}\right)$$

CKM 2005 world average: 0.2262(23) (Blucher)

MILC, Aubin et al, hep-lat/0407028; Heller, this meeting, Bernard LAT05.

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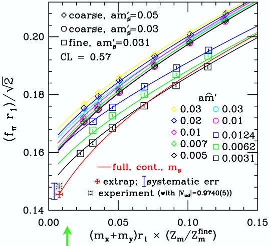

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- fit with 20 free parameters
  m_q^2 terms and many terms from StaggeredChPT

Concerning StaggeredChPT

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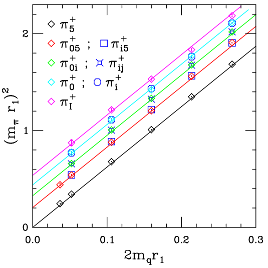


FIG. 9: Squared masses of charged pions for various tastes on the coarse lattices. We use r_1 to set the scale. Tastes that are degenerate by $SO(4)$ symmetry are fit together.

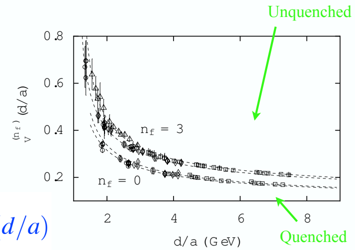
$r_1 \sim 1/600\,\mathrm{MeV}$
$\longrightarrow$ splittings of 450 MeV

SChPT applicable? with the quoted precision?

A recent revolution?

Determining Parameters of QCD : α_s

Combine 3-loop pert. th. for 28 different quantities (mainly Wilson loops) with numerical calc. on lattice



$$W_{latt} = \sum_{n=1}^{3} c_n \alpha_V^n(d/a)$$

Results at 3 values of *a* allows estimates of 4-loop terms. *d/a* is BLM scale - differs for each W, so see running of α_s .

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  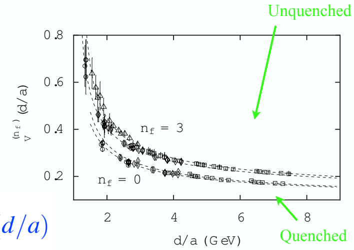

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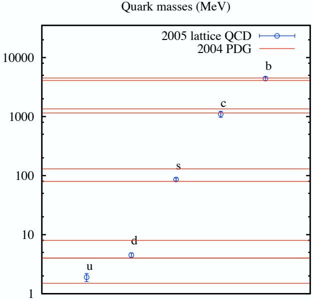
- $\log W_{12} = -5.551\alpha_V(3/a) \times [1 - 0.86\alpha + 1.72\alpha^2 \underbrace{-5(2)}_{\uparrow} \alpha^3 \underbrace{-1(2)}_{\uparrow} \alpha^4 + \ldots]$

  $\alpha = 0.25 \ldots 0.4$ fitted

A recent revolution?

- Determining parameters of QCD: m_q



Use pert. th. to convert lattice bare mass to $\overline{MS}$

New b/c masses (GeV):

$$\overline{m}_b(\overline{m}_b) = 4.4(3)$$

$$\overline{m}_c(\overline{m}_c) = 1.10(13)$$

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Light quark masses now have 2-loop matching!

$\overline{m}_s = 86(5); \overline{m}_u = 1.9(3); \overline{m}_d = 4.5(4)$ in MeV at 2 GeV

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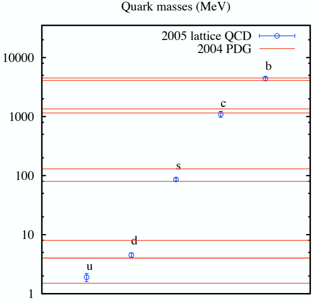

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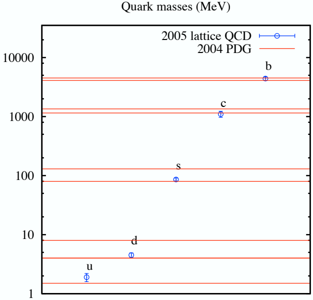

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