A Maple Package for Computing Groebner bases for Linear Recurrence Relations

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Outline

- Janet-like Gröbner Basis algorithm
- Main functions of LFDM
- Timings
- Applications: difference schemes for PDEs, reduction of Feynman integrals

LFDM (Linear Finite Difference Modules)

- implements Janet (-like Gröbner) Bases for linear shift equations
- equations with non-constant coefficients
- can naturally be extended to quasilinear systems
- Applications: difference schemes for PDEs, reduction of Feynman integrals

Main functions of LFDM

`JanetBasis`

compute Janet(-like Gröbner) basis

`InvReduce`

involutive reduction modulo Janet basis

`CompCond`

compatibility conditions (syzygies)

`ResidueClassBasis`, `ShiftRepres`, etc.

residue class basis of factor module, etc.

`HilbertSeries`, `HilbertPolynomial`, etc.

combinatorial devices

`Pol2Shift / Shift2Pol`

conversion between shift operators and eq’s

Algorithm

choose $f \in F$ with the lowest $\mathrm{lm}(f)$ w.r.t. $\prec$

$G := \{f\}$; $Q := F \setminus G$

do

    $h := 0$

    while $Q \neq \emptyset$ and $h = 0$ do

        choose $p \in Q$ with the lowest $\mathrm{lm}(p)$ w.r.t. $\prec$

        $Q := Q \setminus \{p\}$; $h :=$ normal form$(p, G, \prec)$

    if $h \neq 0$ then

        for all $\{g \in G \mid \mathrm{lm}(g) = \theta^{\mu}(\mathrm{lm}(h)), |\mu| > 0\}$ do

            $Q := Q \cup \{g\}$; $G := G \setminus \{g\}$

        $G := G \cup \{h\}$

        $Q := Q \cup \{\, \theta^{\beta} \circ g \mid g \in G,\ \beta \in DP(\mathrm{lm}(g), \mathrm{lm}(G)) \,\}$

until $Q = \emptyset$

return G

Involutive Criteria ***(Gerdt, Blinkov, Yanovich, Apel, Hemmecke)***

For each $p \in G$ let $\mathrm{anc}(p) \in G$ with $\deg(\mathrm{lm}(g))$ minimal s.t. $p = anc(p) \cdot u$ for some monomial u.

Non-multiplicative prolongation $p = x \cdot f$ can be discarded, if for $g \,|_{\mathcal{J}}\, p$ with $f \neq g \in G$ $\prec$-minimal, we have:

$$C_1(p, g) \iff \mathrm{lm}(\mathrm{anc}(p)) \cdot \mathrm{lm}(\mathrm{anc}(g)) = \mathrm{lm}(p)$$

$$C_2(p, g) \iff \mathrm{lcm}(\mathrm{lm}(\mathrm{anc}(p)), \mathrm{lm}(\mathrm{anc}(g))) \mid \mathrm{lm}(p)$$

Involutive Criteria (cont.) *(Gerdt, Blinkov, Yanovich, Apel, Hemmecke)*

Non-multiplicative prolongation $p = x \cdot f$ can be discarded,
if for $g \,|_{\mathcal{J}}\, p$ with $f \neq g \in G$ $\prec$-minimal, we have:

$$C_3(p,g) \;\Leftrightarrow\; \exists\, h \in G :$$

$$\mathrm{lcm}(\mathrm{lm}(h),\, \mathrm{lm}(\mathrm{anc}(p))) \mid \mathrm{lcm}(\mathrm{lm}(\mathrm{anc}(p)),\, \mathrm{lm}(\mathrm{anc}(g)))$$

and

$$\mathrm{lcm}(\mathrm{lm}(h),\, \mathrm{lm}(\mathrm{anc}(g))) \mid \mathrm{lcm}(\mathrm{lm}(\mathrm{anc}(p)),\, \mathrm{lm}(\mathrm{anc}(g)))$$

$C_4(p,g) \;\Leftrightarrow\; \exists\, h \in G,$ non-mult. variable y for h :

$\deg(\mathrm{lm}(h)) = \deg(\mathrm{lm}(f))$ and

$\mathrm{lcm}(\mathrm{lm}(\mathrm{anc}(h)), \mathrm{lm}(\mathrm{anc}(p))) \mid \mathrm{lm}(p)$ and

h precedes f in the list G

Computation of Janet-like Gröbner Basis

`JanetBasis(L, ivar, dvar, ord)`

`L`:	list of (lhs of) shift equations
`ivar` / `dvar`:	list of independent / dependent variables
`ord`:	(optional) term order ($1-4$)

$$\left\{ \begin{array}{l} u(x,y+1)-v(x,y+1) \\ u(x+1,y)-v(x,y) \end{array} \right. \longrightarrow \left\{ \begin{array}{lr} \boxed{u(x,y+1)}-v(x,y+1) & \boxed{* \quad y} \\ \boxed{u(x+1,y)}-v(x,y) & \boxed{x \quad y} \\ v(x,y+1)-\boxed{v(x+1,y+1)} & \boxed{x \quad y} \end{array} \right.$$

`ivar` $=[x,y]$, `dvar` $=[u,v]$, `ord` $=4$ ($\hat{=}$ deg. rev. lex.)

Term orders

1	pure lex.	position over term
2	deg. rev. lex.	position over term
3	pure lex.	term over position
4	deg. rev. lex.	term over position

Ex.: $p := u(x+1,\, y) \;+\; u(x,\, y+2) \;+\; v(x+2,\, y) \;+\; v(x,\, y+3)$

ord	leading shifted term in p
1	$u(x+1,\, y)$
2	$u(x,\, y+2)$
3	$v(x+2,\, y)$
4	$v(x,\, y+3)$

Involutive reduction modulo Janet basis

`InvReduce(p, JB)`

`L`: (lhs of) shift equation to be reduced

`JB`: output of `JanetBasis`

$b_1 = u(x+1,y) - v(x,y)$	$*$	y
$b_2 = u(x,y+1) - v(x,y+1)$	x	y
$b_3 = v(x,y+1) - v(x+1,y+1)$	x	y

$$
\begin{array}{r}
u(x+2,y+2) + \qquad u(x,y) \\
-\theta_x\,\theta_y^2\,b_1 \\
+\theta_y\,b_3 \\
\hline
v(x,y+2) + u(x,y)
\end{array}
$$

Compatibility Conditions

```
> with(LFDM):

> ivar := [x,y]:  dvar := [u]:

> L := [u(x+1,y)-a(x,y), u(x,y+1)-b(x,y)]:

> JanetBasis(L, ivar, dvar);
```

$$[[\mathrm{u}(x,\, y+1) - \mathrm{v}(x,\, y+1),\; \mathrm{u}(x+1,\, y) - \mathrm{v}(x,\, y),\; \mathrm{v}(x,\, y+1) - \mathrm{v}(x+1,\, y+1)],\; [x,\, y],\; [u,\, v]]$$

```
> CompCond(L, ivar, dvar);
```

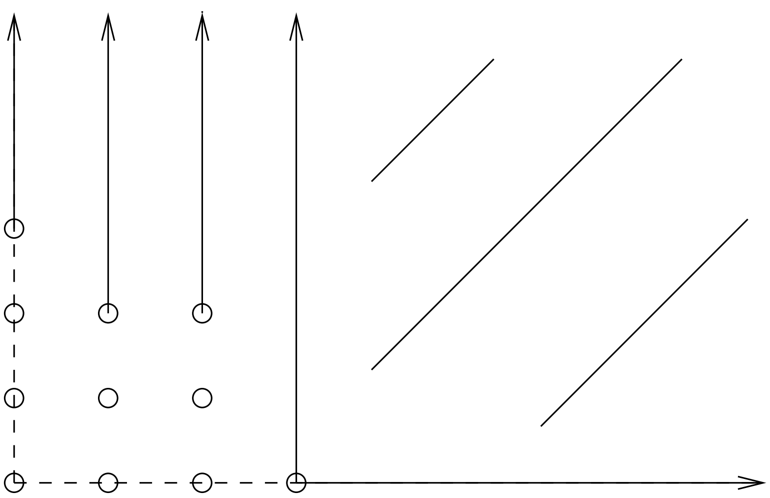
$$[\mathrm{a}(x,\, y+1) - \mathrm{b}(x+1,\, y)]$$

Residue class basis of factor module

`ResidueClassBasis(ivar, dvar)`

`ivar`: list of independent variables

`dvar`: list of dependent variables



$$\longrightarrow \quad \{\, u(x,y),\;\; u(x,y+1),\;\; u(x,y+2),\;\; u(x+1,y),\;\; u(x+1,y+1),\;\; u(x+2,y),\;\; u(x+2,y+1) \,\}$$

Hilbert series

```
> ivar := [x,y]:  dvar := [u]:
```

```
> L := [u(x+3,y),u(x+1,y+2),u(x,y+3)]:
```

```
> JanetBasis(L, ivar, dvar);
```

$$[[\mathrm{u}(x,\, y+3),\, \mathrm{u}(x+1,\, y+2),\, \mathrm{u}(x+3,\, y),\, \mathrm{u}(x+2,\, y+2)],\, [x,\, y],\, [u]]$$

```
> ResidueClassBasis(ivar, dvar);
```

$$[u(x,y), u(x,y+1), u(x+1,y), u(x,y+2),$$
$$u(x+1,y+1), u(x+2,y), u(x+2,y+1)]$$

```
> HilbertSeries(lambda);
```

$$1+2\,\lambda+3\,\lambda^2+\lambda^3$$

Timings

	Groebner	LFDM
cyclic6	68 s	192 s
des18_3	14 s	70 s
eco7	7 s	22 s
eco8	69 s	326 s
extcyc4	1 s	6 s
katsura6	35 s	46 s
noon5	13 s	53 s
redcyc6	4227 s	119 s
reimer5	31 s	159 s

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