Symbolic Summation and Higher Orders in Perturbation Theory

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Introduction

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- Given expression $g(n)$ (depending on n) find expression $f(n)$, such that

$$f(n) \quad = \quad \sum_{i=0}^{n} g(i)$$

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  - Hypergeometric summation
  - Harmonic summation
  - Beyond

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  - Beyond
- Examples in particle physics
- Summary

Polynomial summation

Examples

- Polynomials

$$\sum_{i=0}^{n-1} i = \frac{1}{2}n(n-1)$$

$$\sum_{i=0}^{n-1} i^2 = \frac{1}{6}n(n-1)(2n-1)$$

$$\sum_{i=0}^{n-1} i^3 = \frac{1}{4}n^2(n-1)^2$$

$$\sum_{i=0}^{n-1} i^4 = \frac{1}{30}n(n-1)(2n-1)(3n^2-3n-1)$$

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Rising and falling factorials

- Define rising factorials as $f^{\overline{m}} = f(x) f(x+1) \ldots f(x+m-1)$
  (also known as Pochhammer symbols $(x)_m$)
- Define falling factorials as $f^{\underline{m}} = f(x) f(x-1) \ldots f(x-m+1)$

- Then, with falling factorials

$$\Delta(x^{\underline{m}}) = m x^{\underline{m-1}} \qquad \sum_{i=0}^{n-1} i^{\underline{m}} = \frac{1}{m+1} n^{\underline{m+1}}$$

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- Conversion of polynomial powers x^m
(decomposition with Stirling numbers of second kind $\left\{ \begin{matrix} m \\ i \end{matrix} \right\}$)

$$x^m = \sum_{i=0}^{m} \left\{ \begin{matrix} m \\ i \end{matrix} \right\} x^{\underline{i}}$$

  - Stirling numbers of second kind denote # of ways to partition n things in k non-empty sets

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Examples

$$\sum_{i=0}^{n-1} i \;=\; \sum_{i=0}^{n-1} i^{\underline{1}} \;=\; \frac{1}{2} n^{\underline{2}} \;=\; \frac{1}{2} n(n-1)$$

$$\sum_{i=0}^{n-1} i^2 \;=\; \sum_{i=0}^{n-1} (i^{\underline{2}} + i^{\underline{1}}) \;=\; \frac{1}{3} n^{\underline{3}} + \frac{1}{2} n^{\underline{2}} \;=\; \frac{1}{6} n(n+1)(2n+1)$$

Hypergeometric summation

Definition

- Hypergeometric function ${}_mF_n$

$$
{}_mF_n\left(\begin{array}{c} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{array}\Bigg| z\right) = \sum_{i\geq 0} \frac{a_1^{\overline{i}}\ldots a_m^{\overline{i}}}{b_1^{\overline{i}}\ldots b_n^{\overline{i}}}\,\frac{z^i}{i!}
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$$

Examples

$$
{}_0F_0\left(\,\middle| z\right) = \sum_{i\geq 0} \frac{z^i}{i!} = \mathrm{e}^z
$$

$$
{}_2F_1\left(\begin{array}{c} a,1 \\ 1 \end{array}\middle| z\right) = \sum_{i\geq 0} a^{\overline{i}}\,\frac{z^i}{i!} = \frac{1}{(1-z)^a}
$$

$$
{}_2F_1\left(\begin{array}{c} 1,1 \\ 2 \end{array}\middle| z\right) = z\sum_{i\geq 0} \frac{1^{\overline{i}}1^{\overline{i}}}{2^{\overline{i}}}\,\frac{z^i}{i!} = -\ln(1-z)
$$

Ratios

- A term g_n is hypergeometric, if the ratio $r(n)$ of two consecutive terms is a rational function of n.

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- Example: binomial coefficient

$$\frac{\binom{m}{n+1}}{\binom{m}{n}} = \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(m-n+1)}{\Gamma(n+2)\Gamma(m-n)\Gamma(m)} = \frac{-n+m}{n+1}$$

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- Given a hypergeometric term g, is there hypergeometric term f such that $\Delta f = g$?

$$f_{n+1} - f_n = g_n$$

Gospers algorithm

- Gospers algorithm for indefinite hypergeometric summation determines f_n from a given recursion

$$f_n = f_{n-1} + g_{n-1} = f_{n-2} + g_{n-1} + g_{n-2} = \cdots = f_0 + \sum_{k=0}^{n-1} g_k$$

  - Idea: recursive algorithm; telescoping

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Definite vs. indefinite summation

- Examples

$$\sum_k \binom{n}{k} = \sum_k \binom{n}{k} \qquad\qquad \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Harmonic summation

- Harmonic sums $S_{m_1,\ldots,m_k}(n)$
  Gonzalez-Arroyo, Lopez, Yndurάin '79; Vermaseren '98; Blümlein, Kurth '98; S.M., Uwer, Weinzierl '01; Blümlein '03
  - recursive definition $S_{\pm m_1,\ldots,m_k}(n) = \sum_{i=1}^{n} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$

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- Particle physics
  - dimensional regularization $D = 4 - 2\epsilon$ requires expansion of the Gamma-function around positive integers values ($n \geq 0$)

$$\frac{\Gamma(n+1+\epsilon)}{\Gamma(1+\epsilon)} \quad = \quad \Gamma(n+1)\,\exp\left(-\sum_{k=1}^{\infty} \epsilon^k\, \frac{(-1)^k)}{k}\, S_k(n)\right)$$

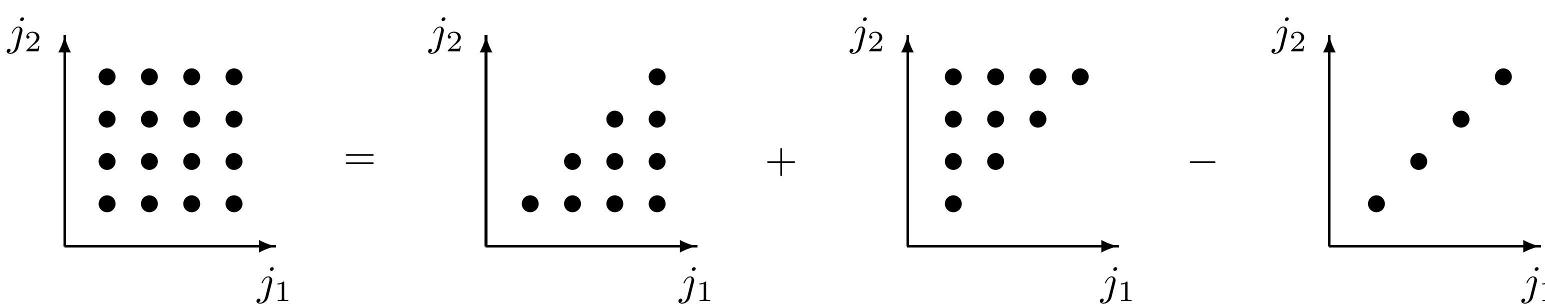
Algorithms for harmonic sums

- Multiplication (Hopf algebra)
  - basic formula (recursion)

$$
\begin{aligned}
S_{m_1,\ldots,m_k}(n) \times S_{m'_1,\ldots,m'_l}(n) \quad = \quad & \sum_{j_1=1}^{n} \frac{1}{j_1^{m_1}} S_{m_2,\ldots,m_k}(j_1) S_{m'_1,\ldots,m'_l}(j_1) \\
& + \sum_{j_2=1}^{n} \frac{1}{j_2^{m'_1}} S_{m_1,\ldots,m_k}(j_2) S_{m'_2,\ldots,m'_l}(j_2) \\
& - \sum_{j=1}^{n} \frac{1}{j^{m_1+m'_1}} S_{m_2,\ldots,m_k}(j) S_{m'_2,\ldots,m'_l}(j)
\end{aligned}
$$

- Proof uses decomposition

$$
\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{i} a_{ij} + \sum_{j=1}^{n}\sum_{i=1}^{j} a_{ij} - \sum_{i=1}^{n} a_{ii}
$$



Algorithms for harmonic sums (cont'd)

- Convolution (sum over $n-j$ and j)

$$\sum_{j=1}^{n-1} \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2,\ldots,n_l}(n-j)$$

- Conjugation

$$-\sum_{j=1}^{n} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j)$$

- Binomial convolution (sum over binomial, $n-j$ and j)

$$-\sum_{j=1}^{n-1} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2,\ldots,n_l}(n-j)$$

Beyond

- Generalized sums $S(n; m_1, ..., m_k; x_1, ..., x_k)$
  - recursive definition

$$S(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} S(i; m_2, ..., m_k; x_2, ..., x_k)$$

  - multiple scales $x_1, ..., x_k$
  - depth k, weight $w = m_1 + ... + m_k$

Example

- Powers of logarithm $\ln(1-x)$

$$\sum_{j=1}^{\infty} \frac{x^j}{j!} \Gamma(j-\epsilon) = \sum_{j=1}^{\infty} \frac{x^j}{j} - \epsilon \sum_{j=1}^{\infty} \frac{x^j}{j} S_1(j-1) + \epsilon^2 \dots$$

$$= -\ln(1-x) - \epsilon \frac{1}{2} \ln(1-x)^2 + \epsilon^2 \dots$$

Harmonic polylogarithms

- Harmonic polylogarithms $H_{m_1,\ldots,m_k}(x)$
  Goncharov '98; Borwein, Bradley,Broadhurst, Lisonek '99; Remiddi, Vermaseren '99
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    - basic functions of lowest weight

      $$H_0(x) = \ln x\,, \quad H_1(x) = -\ln(1-x)\,, \quad H_{-1}(x) = \ln(1+x)$$

    - higher functions defined by recursion

      $$H_{m_1,\dots,m_w}(x) = \int_0^x dz\; f_{m_1}(z)\; H_{m_2,\dots,m_w}(z)$$

      $$f_0(x) = \frac{1}{x}\,, \qquad f_1(x) = \frac{1}{1-x}\,, \qquad f_{-1}(x) = \frac{1}{1+x}$$

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- Harmonic polylogarithms related to generalized sums

  $S(\infty; m_1, ..., m_k; x_1, ..., x_k)$

- (Inverse) Mellin transformation

$$\tilde{f}(N) \;=\; \int_0^1 dx\;\, x^{\,N}\, f(x)$$

  - unique mapping $\dfrac{H_{m_1,\dots,m_w}(x)}{(1 \pm x)} \longleftrightarrow S_{n_1,\dots,n_{w+1}}(N)$

Algorithms for nested sums

- Same structures as for harmonic sums, in particular
  - multiplication

$$S(n; m_1, ...; x_1, ...) \times S(n; m'_1, ...; x'_1, ...)$$

  - convolution
  - conjugation
  - binomial convolution
- Recursive algorithms analogous to harmonic sums solve multiple nested sums

Higher transcendental functions

- Expansion of higher transcendental functions in small parameter
  - expansion parameter ϵ occurs in the rising factorials (Pochhammer symbols)
- Hypergeometric function

$$ {}_2F_1(a,b;c,x_0) = \sum_{i=0}^{\infty} \frac{a^{\overline{i}} b^{\overline{i}}}{c^{\overline{i}}} \frac{x_0^i}{i!} $$

- First Appell function

$$ F_1(a,b_1,b_2;c;x_1,x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c^{\overline{m_1+m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} $$

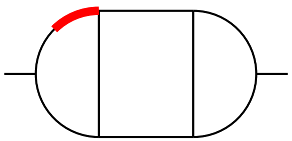
- Second Appell function

$$ F_2(a,b_1,b_2;c_1,c_2;x_1,x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c_1^{\overline{m_1}} c_2^{\overline{m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} $$

Examples in particle physics

Feynman integrals

- Scalar diagram with external momenta P and Q Four-point function with underlying ladder topology



$$= \int \prod_{n}^{3} d^D l_n \, \frac{1}{(P - l_1)^2} \, \frac{1}{l_1^2 \ldots l_8^2}$$

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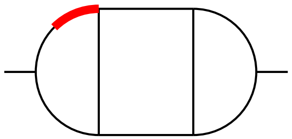
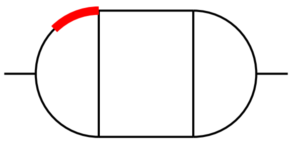


$$= \int \prod_{n}^{3} d^D l_n \, \frac{1}{(P - l_1)^2} \, \frac{1}{l_1^2 \ldots l_8^2}$$

- N-th moment: coefficient of $(2P \cdot Q)^N$



$$= \frac{(2\, P \cdot Q)^N}{(Q^2)^{N+\alpha}} \, C_N$$

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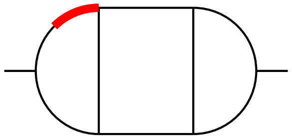

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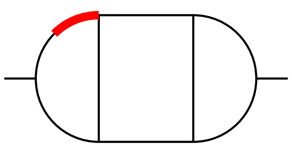

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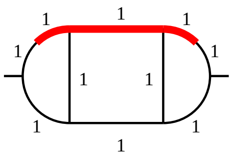
- Taylor expansion

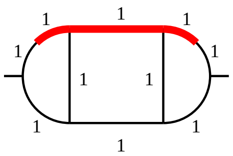
$$\frac{1}{(P - l_1)^2} = \sum_i \frac{(2P \cdot l_1)^i}{(l_1^2)^{i+1}} \quad \longrightarrow \quad \frac{(2P \cdot l_1)^N}{(l_1^2)^N}$$

Difference equations

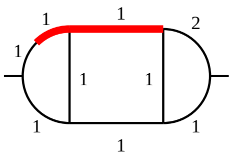
- Single-step difference equation in N
  - extremely simple example



$$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$$



$$+\frac{2}{N+2}$$



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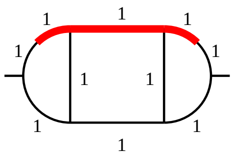

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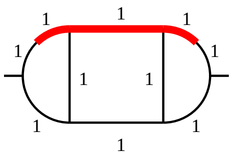

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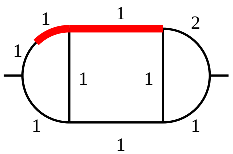

- Formal equation

$$\mathbf{I}(\mathbf{N}) \quad = \quad -\frac{N+3+3\epsilon}{N+2}\,\mathbf{I}(\mathbf{N}-\mathbf{1}) + \frac{2}{N+2}\,\mathbf{G}(\mathbf{N})$$

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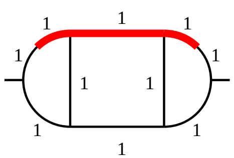

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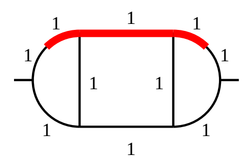

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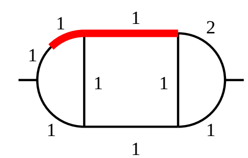

- Formal equation, formal solution

$$\mathbf{I(N)} \quad = \quad (-1)^N \,\frac{\prod_{j=1}^N (j+3+3\epsilon)}{\prod_{j=1}^N (j+2)}\,\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \,\frac{\prod_{j=i+1}^N (j+3+3\epsilon)}{\prod_{j=i}^N (j+2)}\,\mathbf{G(i)}$$

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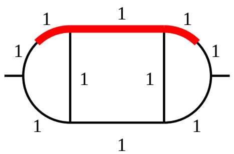

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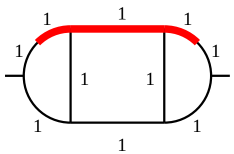

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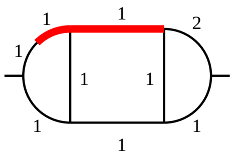


- Formal equation, formal solution, input to solution

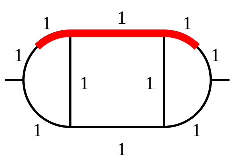
$$\mathbf{I(N)} = (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G(i)}$$

$$\mathbf{I(0)} = -\frac{2}{3}\frac{1}{\epsilon^2} + \frac{23}{3}\frac{1}{\epsilon} - 42$$

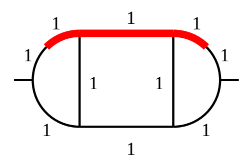
$$\mathbf{G(i)} = \frac{(-1)^i}{\epsilon^2}\frac{2}{3}\left(\frac{S_1(i+2)}{i+2} - \frac{S_{1,2}(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2}\right) + \ldots$$

Difference equations

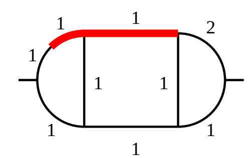
- Single-step difference equation in N
  - extremely simple example



$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$



$+\,\frac{2}{N+2}$



- Formal equation, formal solution, input to solution

$$\mathbf{I(N)} \quad = \quad (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G(i)}$$

$$\mathbf{I(0)} \quad = \quad -\frac{2}{3}\frac{1}{\epsilon^2} + \frac{23}{3}\frac{1}{\epsilon} - 42$$

$$\mathbf{G(i)} \quad = \quad \frac{(-1)^i}{\epsilon^2}\frac{2}{3}\left(\frac{S_1(i+2)}{i+2} - \frac{S_{1,2}(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2}\right) + \ldots$$

- **Upshot**
  - automatic build-up of **nested sums**
  - efficient implementation in FORM

I(N) =

```
sign(N)*ep^-2 * ( 4/3*S(R(1),1 + N)*den(1 + N) + 8/3*S(R(1),1 + N)*den(
  1 + N)^2 + 4/3*S(R(1),2 + N)*den(2 + N) + 4/3*S(R(1),2 + N)*den(2 + N)^2 + 4/3*
  S(R(1),N) + 2/3*S(R(1,2),N) + 2/3*S(R(2),1 + N)*den(1 + N) + 2/3*S(R(2),2 + N)*
  den(2 + N) - 2*S(R(2),N) - 4/3*S(R(2),N)*N + 4*S(R(2,1),N) + 4/3*S(R(2,1),N)*N
   - 6*S(R(3),N) - 2*S(R(3),N)*N - 8/3*den(1 + N)^2 - 4*den(1 + N)^3 - 4/3*den(2
   + N)^2 - 2*den(2 + N)^3 )

+ sign(N)*ep^-1 * (  - 16*S(R(1),1 + N)*den(1 + N) - 88/3*S(R(1),1 + N)*
  den(1 + N)^2 - 20/3*S(R(1),1 + N)*den(1 + N)^3 - 16*S(R(1),2 + N)*den(2 + N) -
  44/3*S(R(1),2 + N)*den(2 + N)^2 - 10/3*S(R(1),2 + N)*den(2 + N)^3 - 20*S(R(1),N)
   + 8/3*S(R(1,1),1 + N)*den(1 + N) + 8/3*S(R(1,1),1 + N)*den(1 + N)^2 + 8/3*S(R(1
  ,1),2 + N)*den(2 + N) + 8/3*S(R(1,1),N) + 10/3*S(R(1,1,2),N) + 10/3*S(R(1,2),1
   + N)*den(1 + N) + 10/3*S(R(1,2),2 + N)*den(2 + N) - 16*S(R(1,2),N) - 4*S(R(1,2)
  ,N)*N + 14*S(R(1,2,1),N) + 4*S(R(1,2,1),N)*N - 24*S(R(1,3),N) - 6*S(R(1,3),N)*N
   - 58/3*S(R(2),1 + N)*den(1 + N) - 40/3*S(R(2),1 + N)*den(1 + N)^2 - 46/3*S(R(2)
  ,2 + N)*den(2 + N) - 6*S(R(2),2 + N)*den(2 + N)^2 + 56/3*S(R(2),N) + 20*S(R(2),N
  )*N + 10*S(R(2,1),1 + N)*den(1 + N) + 6*S(R(2,1),2 + N)*den(2 + N) - 134/3*S(R(2
  ,1),N) - 56/3*S(R(2,1),N)*N + 16/3*S(R(2,1,1),N) + 8/3*S(R(2,1,1),N)*N - 62/3*S(
  R(2,2),N) - 22/3*S(R(2,2),N)*N - 18*S(R(3),1 + N)*den(1 + N) - 12*S(R(3),2 + N)*
  den(2 + N) + 76*S(R(3),N) + 100/3*S(R(3),N)*N - 10*S(R(3,1),N) - 10/3*S(R(3,1),N
  )*N + 36*S(R(4),N) + 12*S(R(4),N)*N + 32*den(1 + N)^2 + 164/3*den(1 + N)^3 + 24*
  den(1 + N)^4 + 16*den(2 + N)^2 + 82/3*den(2 + N)^3 + 12*den(2 + N)^4 )

+ sign(N) * ( 100*S(R(1),1 + N)*den(1 + N) + 168*S(R(1),1 + N)*den(1 + N)
  ^2 + 268/3*S(R(1),1 + N)*den(1 + N)^3 - 16/3*S(R(1),1 + N)*den(1 + N)^4 + 100*S(
  R(1),2 + N)*den(2 + N) + 84*S(R(1),2 + N)*den(2 + N)^2 + 134/3*S(R(1),2 + N)*
  den(2 + N)^3 - 8/3*S(R(1),2 + N)*den(2 + N)^4 + 160*S(R(1),N) - 32*S(R(1,1),1 +
  N)*den(1 + N) - 80/3*S(R(1,1),1 + N)*den(1 + N)^2 - 20/3*S(R(1,1),1 + N)*den(1
   + N)^3 - 32*S(R(1,1),2 + N)*den(2 + N) - 4/3*S(R(1,1),2 + N)*den(2 + N)^2 - 10/
  3*S(R(1,1),2 + N)*den(2 + N)^3 - 40*S(R(1,1),N) + 4/3*S(R(1,1,1),1 + N)*den(1 +
  N) - 40/3*S(R(1,1,1),1 + N)*den(1 + N)^2 + 4/3*S(R(1,1,1),2 + N)*den(2 + N) - 44/
  3*S(R(1,1,1),2 + N)*den(2 + N)^2 + 4/3*S(R(1,1,1),N) + 38/3*S(R(1,1,1,2),N) + 38/
  3*S(R(1,1,2),1 + N)*den(1 + N) + 38/3*S(R(1,1,2),2 + N)*den(2 + N) - 68*S(R(1,1,
  2),N) - 12*S(R(1,1,2),N)*N + 42*S(R(1,1,2,1),N) + 12*S(R(1,1,2,1),N)*N - 76*S(R(
  1,1,3),N) - 18*S(R(1,1,3),N)*N - 170/3*S(R(1,2),1 + N)*den(1 + N) + 40/3*S(R(1,2
  ),1 + N)*den(1 + N)^2 - 134/3*S(R(1,2),2 + N)*den(2 + N) + 14*S(R(1,2),2 + N)*
  den(2 + N)^2 + 430/3*S(R(1,2),N) + 60*S(R(1,2),N)*N + 30*S(R(1,2,1),1 + N)*den(1
   + N) + 18*S(R(1,2,1),2 + N)*den(2 + N) - 452/3*S(R(1,2,1),N) - 56*S(R(1,2,1),N)
  *N + 74/3*S(R(1,2,1,1),N) + 8*S(R(1,2,1,1),N)*N - 248/3*S(R(1,2,2),N) - 22*S(R(1
  ,2,2),N)*N - 58*S(R(1,3),1 + N)*den(1 + N) - 40*S(R(1,3),2 + N)*den(2 + N) + 886/
  3*S(R(1,3),N) + 100*S(R(1,3),N)*N - 116/3*S(R(1,3,1),N) - 10*S(R(1,3,1),N)*N +
  410/3*S(R(1,4),N) + 36*S(R(1,4),N)*N + 186*S(R(2),1 + N)*den(1 + N) + 448/3*S(R(
  2),1 + N)*den(1 + N)^2 + 160/3*S(R(2),1 + N)*den(1 + N)^3 + 138*S(R(2),2 + N)*
  den(2 + N) + 206/3*S(R(2),2 + N)*den(2 + N)^2 + 80/3*S(R(2),2 + N)*den(2 + N)^3
   - 70*S(R(2),N) - 160*S(R(2),N)*N - 338/3*S(R(2,1),1 + N)*den(1 + N) - 64/3*S(R(
  2,1),1 + N)*den(1 + N)^2 - 206/3*S(R(2,1),2 + N)*den(2 + N) - 10/3*S(R(2,1),2 +
  N)*den(2 + N)^2 + 760/3*S(R(2,1),N) + 140*S(R(2,1),N)*N + 50/3*S(R(2,1,1),1 + N)
  *den(1 + N) + 26/3*S(R(2,1,1),2 + N)*den(2 + N) - 170/3*S(R(2,1,1),N) - 100/3*S(
  R(2,1,1),N)*N - 12*S(R(2,1,1,1),N) + 4/3*S(R(2,1,1,1),N)*N + 38/3*S(R(2,1,2),N)
   - 2/3*S(R(2,1,2),N)*N - 182/3*S(R(2,2),1 + N)*den(1 + N) - 116/3*S(R(2,2),2 + N
  )*den(2 + N) + 676/3*S(R(2,2),N) + 308/3*S(R(2,2),N)*N - 118/3*S(R(2,2,1),N) -
  18*S(R(2,2,1),N)*N + 296/3*S(R(2,3),N) + 36*S(R(2,3),N)*N + 694/3*S(R(3),1 + N)*
  den(1 + N) + 188/3*S(R(3),1 + N)*den(1 + N)^2 + 448/3*S(R(3),2 + N)*den(2 + N)
   + 80/3*S(R(3),2 + N)*den(2 + N)^2 - 1454/3*S(R(3),N) - 290*S(R(3),N)*N - 86/3*
  S(R(3,1),1 + N)*den(1 + N) - 56/3*S(R(3,1),2 + N)*den(2 + N) + 440/3*S(R(3,1),N)
   + 164/3*S(R(3,1),N)*N - 10*S(R(3,1,1),N) - 10/3*S(R(3,1,1),N)*N + 80*S(R(3,2),N
  ) + 80/3*S(R(3,2),N)*N + 302/3*S(R(4),1 + N)*den(1 + N) + 194/3*S(R(4),2 + N)*
  den(2 + N) - 434*S(R(4),N) - 556/3*S(R(4),N)*N - 8*S(R(4,1),N) - 8/3*S(R(4,1),N)
  *N - 150*S(R(5),N) - 50*S(R(5),N)*N - 200*den(1 + N)^2 - 380*den(1 + N)^3 - 896/
  3*den(1 + N)^4 - 100*den(1 + N)^5 - 100*den(2 + N)^2 - 190*den(2 + N)^3 - 448/3*
  den(2 + N)^4 - 50*den(2 + N)^5 );
```

I(N) =

```
sign(N)*ep^-2 * ( 4/3*S(R(1),1 + N)*den(1 + N) + 8/3*S(R(1),1 + N)*den(
  1 + N)^2 + 4/3*S(R(1),2 + N)*den(2 + N) + 4/3*S(R(1),2 + N)*den(2 + N)^2 + 4/3*
  S(R(1),N) + 2/3*S(R(1,2),N) + 2/3*S(R(2),1 + N)*den(1 + N) + 2/3*S(R(2),2 + N)*
  den(2 + N) - 2*S(R(2),N) - 4/3*S(R(2),N)*N + 4*S(R(2,1),N) + 4/3*S(R(2,1),N)*N
   - 6*S(R(3),N) - 2*S(R(3),N)*N - 8/3*den(1 + N)^2 - 4*den(1 + N)^3 - 4/3*den(2
   + N)^2 - 2*den(2 + N)^3 )

+ sign(N)*ep^-1 * (  - 16*S(R(1),1 + N)*den(1 + N) - 88/3*S(R(1),1 + N)*
  den(1 + N)^2 - 20/3*S(R(1),1 + N)*den(1 + N)^3 - 16*S(R(1),2 + N)*den(2 + N) -
  44/3*S(R(1),2 + N)*den(2 + N)^2 - 10/3*S(R(1),2 + N)*den(2 + N)^3 - 20*S(R(1),N)
   + 8/3*S(R(1,1),1 + N)*den(1 + N) + 8/3*S(R(1,1),1 + N)*den(1 + N)^2 + 8/3*S(R(1
  ,1),2 + N)*den(2 + N) + 8/3*S(R(1,1),N) + 10/3*S(R(1,1,2),N) + 10/3*S(R(1,2),1
   + N)*den(1 + N) + 10/3*S(R(1,2),2 + N)*den(2 + N) - 16*S(R(1,2),N) - 4*S(R(1,2)
  ,N)*N + 14*S(R(1,2,1),N) + 4*S(R(1,2,1),N)*N - 24*S(R(1,3),N) - 6*S(R(1,3),N)*N
   - 58/3*S(R(2),1 + N)*den(1 + N) - 40/3*S(R(2),1 + N)*den(1 + N)^2 - 46/3*S(R(2)
  ,2 + N)*den(2 + N) - 6*S(R(2),2 + N)*den(2 + N)^2 + 56/3*S(R(2),N) + 20*S(R(2),N
  )*N + 10*S(R(2,1),1 + N)*den(1 + N) + 6*S(R(2,1),2 + N)*den(2 + N) - 134/3*S(R(2
  ,1),N) - 56/3*S(R(2,1),N)*N + 16/3*S(R(2,1,1),N) + 8/3*S(R(2,1,1),N)*N - 62/3*S(
  R(2,2),N) - 22/3*S(R(2,2),N)*N - 18*S(R(3),1 + N)*den(1 + N) - 12*S(R(3),2 + N)*
  den(2 + N) + 76*S(R(3),N) + 100/3*S(R(3),N)*N - 10*S(R(3,1),N) - 10/3*S(R(3,1),N
  )*N + 36*S(R(4),N) + 12*S(R(4),N)*N + 32*den(1 + N)^2 + 164/3*den(1 + N)^3 + 24*
  den(1 + N)^4 + 16*den(2 + N)^2 + 82/3*den(2 + N)^3 + 12*den(2 + N)^4 )

+ sign(N) * ( 100*S(R(1),1 + N)*den(1 + N) + 168*S(R(1),1 + N)*den(1 + N)
  ^2 + 268/3*S(R(1),1 + N)*den(1 + N)^3 - 16/3*S(R(1),1 + N)*den(1 + N)^4 + 100*S(
  R(1),2 + N)*den(2 + N) + 84*S(R(1),2 + N)*den(2 + N)^2 + 134/3*S(R(1),2 + N)*
  den(2 + N)^3 - 8/3*S(R(1),2 + N)*den(2 + N)^4 + 160*S(R(1),N) - 32*S(R(1,1),1 +
  N)*den(1 + N) - 80/3*S(R(1,1),1 + N)*den(1 + N)^2 - 20/3*S(R(1,1),1 + N)*den(1
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  3*S(R(1,1),2 + N)*den(2 + N)^3 - 40*S(R(1,1),N) + 4/3*S(R(1,1,1),1 + N)*den(1 +
  N) - 40/3*S(R(1,1,1),1 + N)*den(1 + N)^2 + 4/3*S(R(1,1,1),2 + N)*den(2 + N) - 44/
  3*S(R(1,1,1),2 + N)*den(2 + N)^2 + 4/3*S(R(1,1,1),N) + 38/3*S(R(1,1,1,2),N) + 38/
  3*S(R(1,1,2),1 + N)*den(1 + N) + 38/3*S(R(1,1,2),2 + N)*den(2 + N) - 68*S(R(1,1,
  2),N) - 12*S(R(1,1,2),N)*N + 42*S(R(1,1,2,1),N) + 12*S(R(1,1,2,1),N)*N - 76*S(R(
  1,1,3),N) - 18*S(R(1,1,3),N)*N - 170/3*S(R(1,2),1 + N)*den(1 + N) + 40/3*S(R(1,2
  ),1 + N)*den(1 + N)^2 - 134/3*S(R(1,2),2 + N)*den(2 + N) + 14*S(R(1,2),2 + N)*
  den(2 + N)^2 + 430/3*S(R(1,2),N) + 60*S(R(1,2),N)*N + 30*S(R(1,2,1),1 + N)*den(1
   + N) + 18*S(R(1,2,1),2 + N)*den(2 + N) - 452/3*S(R(1,2,1),N) - 56*S(R(1,2,1),N)
  *N + 74/3*S(R(1,2,1,1),N) + 8*S(R(1,2,1,1),N)*N - 248/3*S(R(1,2,2),N) - 22*S(R(1
  ,2,2),N)*N - 58*S(R(1,3),1 + N)*den(1 + N) - 40*S(R(1,3),2 + N)*den(2 + N) + 886/
3*S(R(1,3),N) + 100*S(R(1,3),N)*N - 116/3*S(R(1,3,1),N) - 10*S(R(1,3,1),N)*N +
410/3*S(R(1,4),N) + 36*S(R(1,4),N)*N + 186*S(R(2),1 + N)*den(1 + N) + 448/3*S(R(
2),1 + N)*den(1 + N)^2 + 160/3*S(R(2),1 + N)*den(1 + N)^3 + 138*S(R(2),2 + N)*
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 - 70*S(R(2),N) - 160*S(R(2),N)*N - 338/3*S(R(2,1),1 + N)*den(1 + N) - 64/3*S(R(
2,1),1 + N)*den(1 + N)^2 - 206/3*S(R(2,1),2 + N)*den(2 + N) - 10/3*S(R(2,1),2 +
N)*den(2 + N)^2 + 760/3*S(R(2,1),N) + 140*S(R(2,1),N)*N + 50/3*S(R(2,1,1),1 + N)
*den(1 + N) + 26/3*S(R(2,1,1),2 + N)*den(2 + N) - 170/3*S(R(2,1,1),N) - 100/3*S(
R(2,1,1),N)*N - 12*S(R(2,1,1,1),N) + 4/3*S(R(2,1,1,1),N)*N + 38/3*S(R(2,1,2),N)
 - 2/3*S(R(2,1,2),N)*N - 182/3*S(R(2,2),1 + N)*den(1 + N) - 116/3*S(R(2,2),2 + N
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18*S(R(2,2,1),N)*N + 296/3*S(R(2,3),N) + 36*S(R(2,3),N)*N + 694/3*S(R(3),1 + N)*
den(1 + N) + 188/3*S(R(3),1 + N)*den(1 + N)^2 + 448/3*S(R(3),2 + N)*den(2 + N)
 + 80/3*S(R(3),2 + N)*den(2 + N)^2 - 1454/3*S(R(3),N) - 290*S(R(3),N)*N - 86/3*
S(R(3,1),1 + N)*den(1 + N) - 56/3*S(R(3,1),2 + N)*den(2 + N) + 440/3*S(R(3,1),N)
 + 164/3*S(R(3,1),N)*N - 10*S(R(3,1,1),N) - 10/3*S(R(3,1,1),N)*N + 80*S(R(3,2),N
) + 80/3*S(R(3,2),N)*N + 302/3*S(R(4),1 + N)*den(1 + N) + 194/3*S(R(4),2 + N)*
den(2 + N) - 434*S(R(4),N) - 556/3*S(R(4),N)*N - 8*S(R(4,1),N) - 8/3*S(R(4,1),N)
*N - 150*S(R(5),N) - 50*S(R(5),N)*N - 200*den(1 + N)^2 - 380*den(1 + N)^3 - 896/
3*den(1 + N)^4 - 100*den(1 + N)^5 - 100*den(2 + N)^2 - 190*den(2 + N)^3 - 448/3*
den(2 + N)^4 - 50*den(2 + N)^5 );
```

- Result for $\mathbf{I}(\mathbf{N})$ in the G-scheme

Splitting functions in a nut shell

- Calculate anomalous dimensions (Mellin moments of splitting functions) $\longrightarrow$ divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

$$\gamma_{\rm ij}^{(n)}(N) \;=\; -\int_0^1 dx\; x^{\,N-1}\, P_{\rm ij}^{(n)}(x)$$

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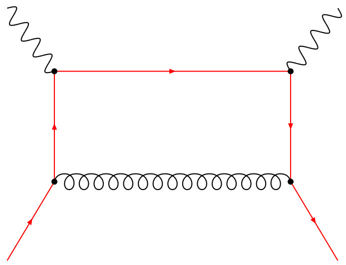
- **One-loop** Feynman diagrams
  $\longrightarrow$ in total 18 for $\gamma_{\rm ij}^{(0)}$ / $P_{\rm ij}^{(0)}$
  (pencil + paper)



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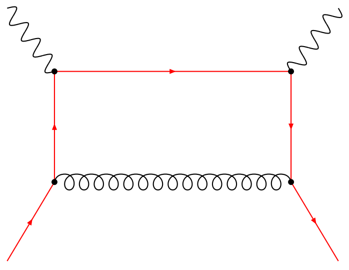
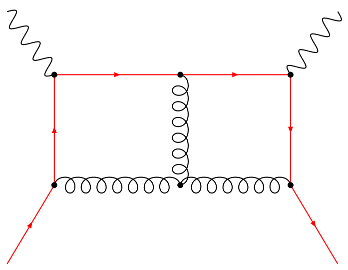
$$\gamma_{\rm ij}^{(n)}(N) \;=\; -\int_0^1 dx\; x^{\,N-1}\, P_{\rm ij}^{(n)}(x)$$

- **One-loop** Feynman diagrams
  $\longrightarrow$ in total 18 for $\gamma_{\rm ij}^{(0)}$ / $P_{\rm ij}^{(0)}$
  (pencil + paper)



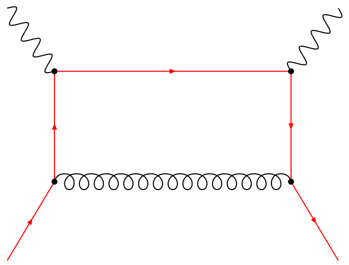
- **Two-loop** Feynman diagrams
  $\longrightarrow$ in total 350 for $\gamma_{\rm ij}^{(1)}$ / $P_{\rm ij}^{(1)}$
  (simple computer algebra)



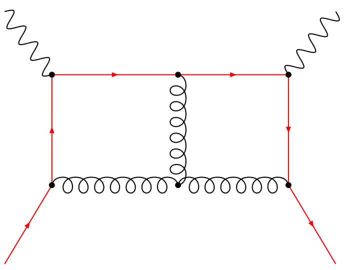
Splitting functions in a nut shell

- Calculate anomalous dimensions (Mellin moments of splitting functions) $\longrightarrow$ divergence of Feynman diagrams in dimensional regularization $D = 4 - 2\epsilon$

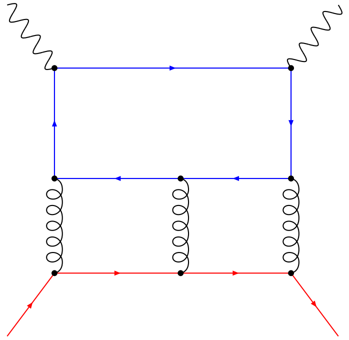
$$\gamma_{\rm ij}^{(n)}(N) \;=\; -\int_0^1 dx\; x^{\,N-1}\, P_{\rm ij}^{(n)}(x)$$



- **One-loop** Feynman diagrams
  $\longrightarrow$ in total 18 for $\gamma_{\rm ij}^{(0)}$ / $P_{\rm ij}^{(0)}$
  (pencil + paper)



- **Two-loop** Feynman diagrams
  $\longrightarrow$ in total 350 for $\gamma_{\rm ij}^{(1)}$ / $P_{\rm ij}^{(1)}$
  (simple computer algebra)



- **Three-loop** Feynman diagrams
  $\longrightarrow$ in total 9607 for $\gamma_{\rm ij}^{(2)}$ / $P_{\rm ij}^{(2)}$
  (cutting edge technology $\longrightarrow$ computer algebra system FORM Vermaseren '89-'04)

LO and NLO splitting functions

$$
\begin{aligned}
P_{\rm ns}^{(0)}(x) &= C_F\,(2p_{\rm qq}(x) + 3\delta(1-x)) \\
P_{\rm ps}^{(0)}(x) &= 0 \\
P_{\rm qg}^{(0)}(x) &= 2n_f\,p_{\rm qg}(x) \\
P_{\rm gq}^{(0)}(x) &= 2C_F\,p_{\rm gq}(x) \\
P_{\rm gg}^{(0)}(x) &= C_A\Big(4p_{\rm gg}(x) + \frac{11}{3}\delta(1-x)\Big) - \frac{2}{3}n_f\,\delta(1-x)
\end{aligned}
$$

$$
\begin{aligned}
P_{\rm ns}^{(1)+}(x) &= 4C_AC_F\Big(p_{\rm qq}(x)\Big[\frac{67}{18} - \zeta_2 + \frac{11}{6}{\rm H}_0 + {\rm H}_{0,0}\Big] + p_{\rm qq}(-x)\Big[\zeta_2 + 2{\rm H}_{-1,0} - {\rm H}_{0,0}\Big] \\
&\quad + \frac{14}{3}(1-x) + \delta(1-x)\Big[\frac{17}{24} + \frac{11}{3}\zeta_2 - 3\zeta_3\Big]\Big) - 4C_Fn_f\Big(p_{\rm qq}(x)\Big[\frac{5}{9} + \frac{1}{3}{\rm H}_0\Big] + \frac{2}{3}(1-x) \\
&\quad + \delta(1-x)\Big[\frac{1}{12} + \frac{2}{3}\zeta_2\Big]\Big) + 4{C_F}^2\Big(2p_{\rm qq}(x)\Big[{\rm H}_{1,0} - \frac{3}{4}{\rm H}_0 + {\rm H}_2\Big] - 2p_{\rm qq}(-x)\Big[\zeta_2 + 2{\rm H}_{-1,0} \\
&\quad - {\rm H}_{0,0}\Big] - (1-x)\Big[1 - \frac{3}{2}{\rm H}_0\Big] - {\rm H}_0 - (1+x){\rm H}_{0,0} + \delta(1-x)\Big[\frac{3}{8} - 3\zeta_2 + 6\zeta_3\Big]\Big) \\
P_{\rm ns}^{(1)-}(x) &= P_{\rm ns}^{(1)+}(x) + 16C_F\Big(C_F - \frac{C_A}{2}\Big)\Big(p_{\rm qq}(-x)\Big[\zeta_2 + 2{\rm H}_{-1,0} - {\rm H}_{0,0}\Big] - 2(1-x) \\
&\quad - (1+x){\rm H}_0\Big)
\end{aligned}
$$

$$
P_{\rm ps}^{(1)}(x) = 4C_Fn_f\Big(\frac{20}{9}\frac{1}{x} - 2 + 6x - 4{\rm H}_0 + x^2\Big[\frac{8}{3}{\rm H}_0 - \frac{56}{9}\Big] + (1+x)\Big[5{\rm H}_0 - 2{\rm H}_{0,0}\Big]\Big)
$$

$$
\begin{aligned}
P_{\rm qg}^{(1)}(x) &= 4C_An_f\Big(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{\rm qg}(-x){\rm H}_{-1,0} - 2p_{\rm qg}(x){\rm H}_{1,1} + x^2\Big[\frac{44}{3}{\rm H}_0 - \frac{218}{9}\Big] \\
&+4(1-x)\Big[{\rm H}_{0,0} - 2{\rm H}_0 + x{\rm H}_1\Big] - 4\zeta_2x - 6{\rm H}_{0,0} + 9{\rm H}_0\Big) + 4C_Fn_f\Big(2p_{\rm qg}(x)\Big[{\rm H}_{1,0} + {\rm H}_{1,1} + {\rm H}_2 \\
&-\zeta_2\Big] + 4x^2\Big[{\rm H}_0 + {\rm H}_{0,0} + \frac{5}{2}\Big] + 2(1-x)\Big[{\rm H}_0 + {\rm H}_{0,0} - 2x{\rm H}_1 + \frac{29}{4}\Big] - \frac{15}{2} - {\rm H}_{0,0} - \frac{1}{2}{\rm H}_0\Big)
\end{aligned}
$$

$$
\begin{aligned}
P_{\rm gq}^{(1)}(x) &= 4C_AC_F\Big(\frac{1}{x} + 2p_{\rm gq}(x)\Big[{\rm H}_{1,0} + {\rm H}_{1,1} + {\rm H}_2 - \frac{11}{6}{\rm H}_1\Big] - x^2\Big[\frac{8}{3}{\rm H}_0 - \frac{44}{9}\Big] + 4\zeta_2 - 2 \\
&-7{\rm H}_0 + 2{\rm H}_{0,0} - 2{\rm H}_1x + (1+x)\Big[2{\rm H}_{0,0} - 5{\rm H}_0 + \frac{37}{9}\Big] - 2p_{\rm gq}(-x){\rm H}_{-1,0}\Big) - 4C_Fn_f\Big(\frac{2}{3}x \\
&-p_{\rm gq}(x)\Big[\frac{2}{3}{\rm H}_1 - \frac{10}{9}\Big]\Big) + 4{C_F}^2\Big(p_{\rm gq}(x)\Big[3{\rm H}_1 - 2{\rm H}_{1,1}\Big] + (1+x)\Big[{\rm H}_{0,0} - \frac{7}{2} + \frac{7}{2}{\rm H}_0\Big] - 3{\rm H}_{0,0} \\
&+1 - \frac{3}{2}{\rm H}_0 + 2{\rm H}_1x\Big)
\end{aligned}
$$

$$
\begin{aligned}
P_{\rm gg}^{(1)}(x) &= 4C_An_f\Big(1 - x - \frac{10}{9}p_{\rm gg}(x) - \frac{13}{9}\Big(\frac{1}{x} - x^2\Big) - \frac{2}{3}(1+x){\rm H}_0 - \frac{2}{3}\delta(1-x)\Big) + 4{C_A}^2\Big(27 \\
&+(1+x)\Big[\frac{11}{3}{\rm H}_0 + 8{\rm H}_{0,0} - \frac{27}{2}\Big] + 2p_{\rm gg}(-x)\Big[{\rm H}_{0,0} - 2{\rm H}_{-1,0} - \zeta_2\Big] - \frac{67}{9}\Big(\frac{1}{x} - x^2\Big) - 12{\rm H}_0 \\
&-\frac{44}{3}x^2{\rm H}_0 + 2p_{\rm gg}(x)\Big[\frac{67}{18} - \zeta_2 + {\rm H}_{0,0} + 2{\rm H}_{1,0} + 2{\rm H}_2\Big] + \delta(1-x)\Big[\frac{8}{3} + 3\zeta_3\Big]\Big) + 4C_Fn_f\Big(2{\rm H}_0 \\
&+\frac{2}{3}\frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x)\Big[4 - 5{\rm H}_0 - 2{\rm H}_{0,0}\Big] - \frac{1}{2}\delta(1-x)\Big)\ .
\end{aligned}
$$

NNLO non-singlet splitting functions

S.M., Vermaseren, Vogt ‘04

$$
\begin{aligned}
P_{\rm ns}^{(2)+}(x) \; = \; & 16 C_A C_F n_f \Big(\frac{1}{6} p_{\rm qq}(x) \Big[\frac{10}{3}\zeta_2 - \frac{209}{36} - 9\zeta_3 - \frac{167}{18}\mathrm{H}_0 + 2\mathrm{H}_0\zeta_2 - 7\mathrm{H}_{0,0} - 2\mathrm{H}_{0,0,0} \\
& + 3\mathrm{H}_{1,0,0} - \mathrm{H}_3 \Big] + \frac{1}{3} p_{\rm qq}(-x) \Big[\frac{3}{2}\zeta_3 - \frac{5}{3}\zeta_2 - \mathrm{H}_{-2,0} - 2\mathrm{H}_{-1}\zeta_2 - \frac{10}{3}\mathrm{H}_{-1,0} - \mathrm{H}_{-1,0,0} \\
& + 2\mathrm{H}_{-1,2} + \frac{1}{2}\mathrm{H}_0\zeta_2 + \frac{5}{3}\mathrm{H}_{0,0} + \mathrm{H}_{0,0,0} - \mathrm{H}_3 \Big] + (1-x) \Big[\frac{1}{6}\zeta_2 - \frac{257}{54} - \frac{43}{18}\mathrm{H}_0 - \frac{1}{6}\mathrm{H}_{0,0} - \mathrm{H}_1 \Big] \\
& - (1+x) \Big[\frac{2}{3}\mathrm{H}_{-1,0} + \frac{1}{2}\mathrm{H}_2 \Big] + \frac{1}{3}\zeta_2 + \mathrm{H}_0 + \frac{1}{6}\mathrm{H}_{0,0} + \delta(1-x) \Big[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}\zeta_2^{\,2} + \frac{25}{18}\zeta_3 \Big] \Big) \\
& + 16 C_A {C_F}^2 \Big(p_{\rm qq}(x) \Big[\frac{5}{6}\zeta_3 - \frac{69}{20}\zeta_2^{\,2} - \mathrm{H}_{-3,0} - 3\mathrm{H}_{-2}\zeta_2 - 14\mathrm{H}_{-2,-1,0} + 3\mathrm{H}_{-2,0} + 5\mathrm{H}_{-2,0,0} \\
& - 4\mathrm{H}_{-2,2} - \frac{151}{48}\mathrm{H}_0 + \frac{41}{12}\mathrm{H}_0\zeta_2 - \frac{17}{2}\mathrm{H}_0\zeta_3 - \frac{13}{4}\mathrm{H}_{0,0} - 4\mathrm{H}_{0,0}\zeta_2 - \frac{23}{12}\mathrm{H}_{0,0,0} + 5\mathrm{H}_{0,0,0,0} + \frac{2}{3}\mathrm{H}_3 \\
& - 24\mathrm{H}_1\zeta_3 - 16\mathrm{H}_{1,-2,0} + \frac{67}{9}\mathrm{H}_{1,0} - 2\mathrm{H}_{1,0}\zeta_2 + \frac{31}{3}\mathrm{H}_{1,0,0} + 11\mathrm{H}_{1,0,0,0} + 8\mathrm{H}_{1,1,0,0} - 8\mathrm{H}_{1,3} + \mathrm{H}_4 \\
& + \frac{67}{9}\mathrm{H}_2 - 2\mathrm{H}_2\zeta_2 + \frac{11}{3}\mathrm{H}_{2,0} + 5\mathrm{H}_{2,0,0} + \mathrm{H}_{3,0} \Big] + p_{\rm qq}(-x) \Big[\frac{1}{4}\zeta_2^{\,2} - \frac{67}{9}\zeta_2 + \frac{31}{4}\zeta_3 + 5\mathrm{H}_{-3,0} \\
& - 32\mathrm{H}_{-2}\zeta_2 - 4\mathrm{H}_{-2,-1,0} - \frac{31}{6}\mathrm{H}_{-2,0} + 21\mathrm{H}_{-2,0,0} + 30\mathrm{H}_{-2,2} - \frac{31}{3}\mathrm{H}_{-1}\zeta_2 - 42\mathrm{H}_{-1}\zeta_3 + \frac{9}{4}\mathrm{H}_0 \\
& - 4\mathrm{H}_{-1,-2,0} + 56\mathrm{H}_{-1,-1}\zeta_2 - 36\mathrm{H}_{-1,-1,0,0} - 56\mathrm{H}_{-1,-1,2} - \frac{134}{9}\mathrm{H}_{-1,0} - 42\mathrm{H}_{-1,0}\zeta_2 - \mathrm{H}_{3,0} \\
& + 32\mathrm{H}_{-1,3} - \frac{31}{6}\mathrm{H}_{-1,0,0} + 17\mathrm{H}_{-1,0,0,0} + \frac{31}{3}\mathrm{H}_{-1,2} + 2\mathrm{H}_{-1,2,0} + \frac{13}{12}\mathrm{H}_0\zeta_2 + \frac{29}{2}\mathrm{H}_0\zeta_3 + \frac{67}{9}\mathrm{H}_{0,0} \\
& + 13\mathrm{H}_{0,0}\zeta_2 + \frac{89}{12}\mathrm{H}_{0,0,0} - 5\mathrm{H}_{0,0,0,0} - 7\mathrm{H}_2\zeta_2 - \frac{31}{6}\mathrm{H}_3 - 10\mathrm{H}_4 \Big] + (1-x) \Big[\frac{133}{36} + 4\mathrm{H}_{0,0,0,0} \\
& - \frac{167}{4}\zeta_3 - 2\mathrm{H}_0\zeta_3 - 2\mathrm{H}_{-3,0} + \mathrm{H}_{-2}\zeta_2 + 2\mathrm{H}_{-2,-1,0} - 3\mathrm{H}_{-2,0,0} + \frac{77}{4}\mathrm{H}_{0,0,0} - \frac{209}{6}\mathrm{H}_1 - 7\mathrm{H}_1\zeta_2 \\
& + 4\mathrm{H}_{1,0,0} + \frac{14}{3}\mathrm{H}_{1,0} \Big] + (1+x) \Big[\frac{43}{2}\zeta_2 - 3\zeta_2^{\,2} + \frac{25}{2}\mathrm{H}_{-2,0} - 31\mathrm{H}_{-1}\zeta_2 - 14\mathrm{H}_{-1,-1,0} - \frac{13}{3}\mathrm{H}_{-1,0} \\
& + 24\mathrm{H}_{-1,2} + 23\mathrm{H}_{-1,0,0} + \frac{55}{2}\mathrm{H}_0\zeta_2 + 5\mathrm{H}_{0,0}\zeta_2 + \frac{1457}{48}\mathrm{H}_0 - \frac{1025}{36}\mathrm{H}_{0,0} - \frac{155}{6}\mathrm{H}_2 + \mathrm{H}_2\zeta_2 - 15\mathrm{H}_3 \\
& + 2\mathrm{H}_{2,0,0} - 3\mathrm{H}_4 \Big] - 5\zeta_2 - \frac{1}{2}\zeta_2^{\,2} + 50\zeta_3 - 2\mathrm{H}_{-3,0} - 7\mathrm{H}_{-2,0} - \mathrm{H}_0\zeta_3 - \frac{37}{2}\mathrm{H}_0\zeta_2 - \frac{242}{9}\mathrm{H}_0 \\
& - 2\mathrm{H}_{0,0}\zeta_2 + \frac{185}{6}\mathrm{H}_{0,0} - 22\mathrm{H}_{0,0,0} - 4\mathrm{H}_{0,0,0,0} + \frac{28}{3}\mathrm{H}_2 + 6\mathrm{H}_3 + \delta(1-x) \Big[\frac{151}{64} + \zeta_2\zeta_3 - \frac{205}{24}\zeta_2 \\
& - \frac{247}{60}\zeta_2^{\,2} + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5 \Big] \Big) + 16 {C_A}^2 C_F \Big(p_{\rm qq}(x) \Big[\frac{245}{48} - \frac{67}{18}\zeta_2 + \frac{12}{5}\zeta_2^{\,2} + \frac{1}{2}\zeta_3 + \frac{1043}{216}\mathrm{H}_0 \\
& + \mathrm{H}_{-3,0} + 4\mathrm{H}_{-2,-1,0} - \frac{3}{2}\mathrm{H}_{-2,0} - \mathrm{H}_{-2,0,0} + 2\mathrm{H}_{-2,2} - \frac{31}{12}\mathrm{H}_0\zeta_2 + 4\mathrm{H}_0\zeta_3 + \frac{389}{72}\mathrm{H}_{0,0} - 2\mathrm{H}_{2,0,0} \\
& - \mathrm{H}_{0,0,0,0} + 9\mathrm{H}_1\zeta_3 + 6\mathrm{H}_{1,-2,0} - \mathrm{H}_{1,0}\zeta_2 - \frac{11}{4}\mathrm{H}_{1,0,0} - 3\mathrm{H}_{1,0,0,0} - 4\mathrm{H}_{1,1,0,0} + 4\mathrm{H}_{1,3} + \frac{31}{12}\mathrm{H}_{0,0,0} \\
& + \frac{11}{12}\mathrm{H}_3 + \mathrm{H}_4 \Big] + p_{\rm qq}(-x) \Big[\frac{67}{18}\zeta_2 - \zeta_2^{\,2} - \frac{11}{4}\zeta_3 - \mathrm{H}_{-3,0} + 8\mathrm{H}_{-2}\zeta_2 + \frac{11}{6}\mathrm{H}_{-2,0} - 4\mathrm{H}_{-2,0,0} \\
& - 3\mathrm{H}_{-1,0,0,0} + \frac{11}{3}\mathrm{H}_{-1}\zeta_2 + 12\mathrm{H}_{-1}\zeta_3 - 16\mathrm{H}_{-1,-1}\zeta_2 + 8\mathrm{H}_{-1,-1,0,0} + 16\mathrm{H}_{-1,-1,2} + \frac{67}{9}\mathrm{H}_{-1,0}
\end{aligned}
$$

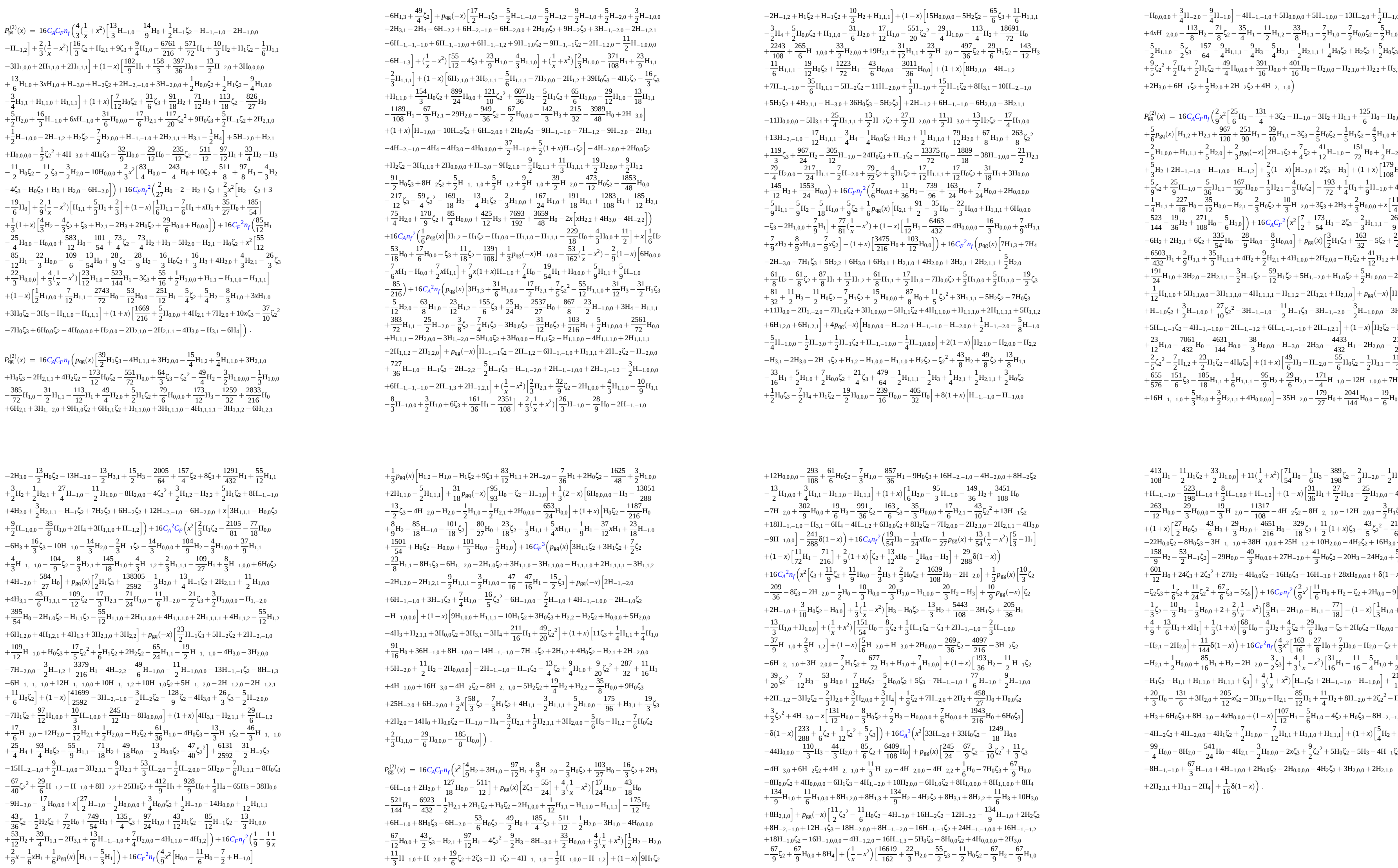
$$
\begin{aligned}
& - 8\mathrm{H}_{-2,2} + 11\mathrm{H}_{-1,0}\zeta_2 + \frac{11}{6}\mathrm{H}_{-1,0,0} - \frac{11}{3}\mathrm{H}_{-1,2} - 8\mathrm{H}_{-1,3} - \frac{3}{4}\mathrm{H}_0 - \frac{1}{6}\mathrm{H}_0\zeta_2 - 4\mathrm{H}_0\zeta_3 - \frac{67}{18}\mathrm{H}_{0,0} \\
& - 3\mathrm{H}_{0,0}\zeta_2 - \frac{31}{12}\mathrm{H}_{0,0,0} + \mathrm{H}_{0,0,0,0} + 2\mathrm{H}_2\zeta_2 + \frac{11}{6}\mathrm{H}_3 + 2\mathrm{H}_4 \Big] + (1-x) \Big[\frac{1883}{108} - \frac{1}{2}\mathrm{H}_{0,0,0,0} + 11\mathrm{H}_1 \\
& - \mathrm{H}_{-2,-1,0} + \frac{1}{2}\mathrm{H}_{-3,0} - \frac{1}{2}\mathrm{H}_{-2}\zeta_2 + \frac{1}{2}\mathrm{H}_{-2,0,0} + \frac{523}{36}\mathrm{H}_0 + \mathrm{H}_0\zeta_3 - \frac{13}{3}\mathrm{H}_{0,0} - \frac{5}{2}\mathrm{H}_{0,0,0} + 2\mathrm{H}_1\zeta_2 \\
& - 2\mathrm{H}_{1,0,0} \Big] + (1+x) \Big[8\mathrm{H}_{-1}\zeta_2 + 4\mathrm{H}_{-1,-1,0} + \frac{8}{3}\mathrm{H}_{-1,0} - 5\mathrm{H}_{-1,0,0} - 6\mathrm{H}_{-1,2} - \frac{13}{3}\zeta_2 + \frac{3}{8}\zeta_2^{\,2} \\
& - \frac{43}{4}\zeta_3 - \frac{5}{2}\mathrm{H}_{-2,0} - \frac{11}{2}\mathrm{H}_0\zeta_2 - \frac{1}{2}\mathrm{H}_2\zeta_2 - \frac{5}{4}\mathrm{H}_{0,0}\zeta_2 + 7\mathrm{H}_2 - \frac{1}{4}\mathrm{H}_{2,0,0} + 3\mathrm{H}_3 + \frac{3}{4}\mathrm{H}_4 \Big] + \frac{1}{2}\mathrm{H}_{0,0}\zeta_2 \\
& + \frac{1}{4}\zeta_2^{\,2} - \frac{8}{3}\zeta_2 + \frac{17}{2}\zeta_3 + \mathrm{H}_{-2,0} - \frac{19}{2}\mathrm{H}_0 + \frac{5}{2}\mathrm{H}_0\zeta_2 - \mathrm{H}_0\zeta_3 + \frac{13}{3}\mathrm{H}_{0,0} + \frac{5}{2}\mathrm{H}_{0,0,0} + \frac{1}{2}\mathrm{H}_{0,0,0,0} \\
& - \delta(1-x) \Big[\frac{1657}{576} - \frac{281}{27}\zeta_2 + \frac{1}{8}\zeta_2^{\,2} + \frac{97}{9}\zeta_3 - \frac{5}{2}\zeta_5 \Big] \Big) + 16 C_F n_f^2 \Big(\frac{1}{18} p_{\rm qq}(x) \Big[\mathrm{H}_{0,0} - \frac{1}{3} + \frac{5}{3}\mathrm{H}_0 \Big] \\
& + (1-x) \Big[\frac{13}{54} + \frac{1}{9}\mathrm{H}_0 \Big] - \delta(1-x) \Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 \Big] \Big) + 16 {C_F}^2 n_f \Big(\frac{1}{3} p_{\rm qq}(x) \Big[5\zeta_3 - 4\mathrm{H}_{1,0,0} \\
& - \frac{55}{16} + \frac{5}{8}\mathrm{H}_0 + \mathrm{H}_0\zeta_2 + \frac{3}{2}\mathrm{H}_{0,0} - \mathrm{H}_{0,0,0} - \frac{10}{3}\mathrm{H}_{1,0} - \frac{10}{3}\mathrm{H}_2 - 2\mathrm{H}_{2,0} - 2\mathrm{H}_3 \Big] + \frac{2}{3} p_{\rm qq}(-x) \Big[\frac{5}{3}\zeta_2 \\
& - \frac{3}{2}\zeta_3 + \mathrm{H}_{-2,0} + 2\mathrm{H}_{-1}\zeta_2 + \frac{10}{3}\mathrm{H}_{-1,0} + \mathrm{H}_{-1,0,0} - 2\mathrm{H}_{-1,2} - \frac{1}{2}\mathrm{H}_0\zeta_2 - \frac{5}{3}\mathrm{H}_{0,0} - \mathrm{H}_{0,0,0} + \mathrm{H}_3 \Big] \\
& - (1-x) \Big[\frac{10}{9} + \frac{19}{18}\mathrm{H}_{0,0} - \frac{4}{3}\mathrm{H}_1 + \frac{2}{3}\mathrm{H}_{1,0} + \frac{4}{3}\mathrm{H}_2 \Big] + (1+x) \Big[\frac{4}{3}\mathrm{H}_{-1,0} - \frac{25}{24}\mathrm{H}_0 + \frac{1}{2}\mathrm{H}_{0,0,0} \Big] + \frac{2}{9}\mathrm{H}_0 \\
& + \frac{7}{9}\mathrm{H}_{0,0} + \frac{4}{3}\mathrm{H}_2 - \delta(1-x) \Big[\frac{23}{16} - \frac{5}{12}\zeta_2 - \frac{29}{30}\zeta_2^{\,2} + \frac{17}{6}\zeta_3 \Big] \Big) + 16 {C_F}^3 \Big(p_{\rm qq}(x) \Big[\frac{9}{10}\zeta_2^{\,2} - 2\mathrm{H}_{-3,0} \\
& + 6\mathrm{H}_{-2}\zeta_2 + 12\mathrm{H}_{-2,-1,0} - 6\mathrm{H}_{-2,0,0} - \frac{3}{16}\mathrm{H}_0 - \frac{3}{2}\mathrm{H}_0\zeta_2 + \mathrm{H}_0\zeta_3 + \frac{13}{8}\mathrm{H}_{0,0} - 2\mathrm{H}_{0,0,0,0} + 8\mathrm{H}_{1,3} \\
& + 12\mathrm{H}_1\zeta_3 + 8\mathrm{H}_{1,-2,0} - 6\mathrm{H}_{1,0,0} - 4\mathrm{H}_{1,0,0,0} + 4\mathrm{H}_{1,2,0} - 3\mathrm{H}_{2,0} + 2\mathrm{H}_{2,0,0} + 4\mathrm{H}_{2,1,0} + 4\mathrm{H}_{2,2} \\
& + 4\mathrm{H}_{3,0} + 4\mathrm{H}_{3,1} + 2\mathrm{H}_4 \Big] + p_{\rm qq}(-x) \Big[\frac{7}{2}\zeta_2^{\,2} - \frac{9}{2}\zeta_3 - 6\mathrm{H}_{-3,0} + 32\mathrm{H}_{-2}\zeta_2 + 8\mathrm{H}_{-2,-1,0} + 3\mathrm{H}_{-2,0} \\
& - 26\mathrm{H}_{-2,0,0} - 28\mathrm{H}_{-2,2} + 6\mathrm{H}_{-1}\zeta_2 + 36\mathrm{H}_{-1}\zeta_3 + 8\mathrm{H}_{-1,-2,0} - 48\mathrm{H}_{-1,-1}\zeta_2 + 40\mathrm{H}_{-1,-1,0,0} \\
& + 48\mathrm{H}_{-1,-1,2} + 40\mathrm{H}_{-1,0}\zeta_2 + 3\mathrm{H}_{-1,0,0} - 22\mathrm{H}_{-1,0,0,0} - 6\mathrm{H}_{-1,2} - 4\mathrm{H}_{-1,2,0} - 32\mathrm{H}_{-1,3} - \frac{3}{2}\mathrm{H}_0 \\
& - \frac{3}{2}\mathrm{H}_0\zeta_2 - 13\mathrm{H}_0\zeta_3 - 14\mathrm{H}_{0,0}\zeta_2 - \frac{9}{2}\mathrm{H}_{0,0,0} + 6\mathrm{H}_{0,0,0,0} + 6\mathrm{H}_2\zeta_2 + 3\mathrm{H}_3 + 2\mathrm{H}_{3,0} + 12\mathrm{H}_4 \Big] \\
& + (1-x) \Big[2\mathrm{H}_{-3,0} - \frac{31}{8} + 4\mathrm{H}_{-2,0,0} + \mathrm{H}_{0,0}\zeta_2 - 3\mathrm{H}_{0,0,0,0} + 35\mathrm{H}_1 + 6\mathrm{H}_1\zeta_2 - \mathrm{H}_{1,0} + \frac{5}{2}\mathrm{H}_{2,0} \Big] \\
& + (1+x) \Big[\frac{37}{10}\zeta_2^{\,2} - \frac{93}{4}\zeta_2 - \frac{81}{2}\zeta_3 - 15\mathrm{H}_{-2,0} + 30\mathrm{H}_{-1}\zeta_2 + 12\mathrm{H}_{-1,-1,0} - 2\mathrm{H}_{-1,0} - 26\mathrm{H}_{-1,0,0} \\
& - 24\mathrm{H}_{-1,2} - \frac{539}{16}\mathrm{H}_0 - 28\mathrm{H}_0\zeta_2 + \frac{191}{8}\mathrm{H}_{0,0} + 20\mathrm{H}_{0,0,0} + \frac{85}{4}\mathrm{H}_2 - 3\mathrm{H}_{2,0,0} - 2\mathrm{H}_{3,0} + 13\mathrm{H}_3 \\
& - \mathrm{H}_4 \Big] + 4\zeta_2 + 33\zeta_3 + 4\mathrm{H}_{-3,0} + 10\mathrm{H}_{-2,0} + \frac{67}{2}\mathrm{H}_0 + 6\mathrm{H}_0\zeta_3 + 19\mathrm{H}_0\zeta_2 - 25\mathrm{H}_{0,0} - 17\mathrm{H}_{0,0,0} \\
& - 2\mathrm{H}_2 - \mathrm{H}_{2,0} - 4\mathrm{H}_3 + \delta(1-x) \Big[\frac{29}{32} - 2\zeta_2\zeta_3 + \frac{9}{8}\zeta_2 + \frac{18}{5}\zeta_2^{\,2} + \frac{17}{4}\zeta_3 - 15\zeta_5 \Big] \Big) \; .
\end{aligned}
$$

$$
\begin{aligned}
P_{\rm ns}^{(2)-}(x) \; = \; & P_{\rm ns}^{(2)+}(x) + 16 C_A C_F \Big(C_F - \frac{C_A}{2} \Big) \Big(p_{\rm qq}(-x) \Big[\frac{134}{9}\zeta_2 - 4\zeta_2^{\,2} - 11\zeta_3 - 4\mathrm{H}_{-3,0} \\
& + 32\mathrm{H}_{-2}\zeta_2 + \frac{22}{3}\mathrm{H}_{-2,0} - 16\mathrm{H}_{-2,0,0} - 32\mathrm{H}_{-2,2} + \frac{44}{3}\mathrm{H}_{-1}\zeta_2 + 48\mathrm{H}_{-1}\zeta_3 - 64\mathrm{H}_{-1,-1}\zeta_2 \\
& + 32\mathrm{H}_{-1,-1,0,0} + 64\mathrm{H}_{-1,-1,2} + \frac{268}{9}\mathrm{H}_{-1,0} + 44\mathrm{H}_{-1,0}\zeta_2 + \frac{22}{3}\mathrm{H}_{-1,0,0} - 12\mathrm{H}_{-1,0,0,0} - \frac{44}{3}\mathrm{H}_{-1,2} \\
& - 32\mathrm{H}_{-1,3} - 3\mathrm{H}_0 - \frac{2}{3}\mathrm{H}_0\zeta_2 - 16\mathrm{H}_0\zeta_3 - \frac{134}{9}\mathrm{H}_{0,0} - 12\mathrm{H}_{0,0}\zeta_2 - \frac{31}{3}\mathrm{H}_{0,0,0} + 4\mathrm{H}_{0,0,0,0} + 8\mathrm{H}_2\zeta_2 \\
& + \frac{22}{3}\mathrm{H}_3 + 8\mathrm{H}_4 \Big] + (1-x) \Big[\frac{367}{18} + \frac{1}{2}\zeta_2^{\,2} + 2\mathrm{H}_{-3,0} - 2\mathrm{H}_{-2}\zeta_2 - 4\mathrm{H}_{-2,-1,0} - 10\mathrm{H}_{-2,0} - 2\mathrm{H}_{0,0} \\
& + 2\mathrm{H}_{-2,0,0} + 2\mathrm{H}_0\zeta_3 + \mathrm{H}_{0,0}\zeta_2 - \mathrm{H}_{0,0,0,0} + 8\mathrm{H}_1\zeta_2 + \frac{140}{3}\mathrm{H}_1 \Big] + (1+x) \Big[32\mathrm{H}_{-1}\zeta_2 - 18\zeta_2 \\
& - 23\zeta_3 + \frac{26}{3}\mathrm{H}_{-1,0} - 16\mathrm{H}_{-1,0,0} - 32\mathrm{H}_{-1,2} - \frac{481}{18}\mathrm{H}_0 - 29\mathrm{H}_0\zeta_2 + 5\mathrm{H}_{0,0,0} + 24\mathrm{H}_3 + \frac{70}{3}\mathrm{H}_2 \Big] \\
& - 2\zeta_2 - 2\zeta_3 + 32\mathrm{H}_0 + 14\mathrm{H}_0\zeta_2 + 2\mathrm{H}_{0,0,0} - 16\mathrm{H}_3 \Big) + 16 C_F n_f \Big(C_F - \frac{C_A}{2} \Big) \Big(p_{\rm qq}(-x) \Big[2\zeta_3 \\
& - \frac{20}{9}\zeta_2 - \frac{4}{3}\mathrm{H}_{-2,0} - \frac{8}{3}\mathrm{H}_{-1}\zeta_2 - \frac{40}{9}\mathrm{H}_{-1,0} - \frac{4}{3}\mathrm{H}_{-1,0,0} + \frac{8}{3}\mathrm{H}_{-1,2} + \frac{2}{3}\mathrm{H}_0\zeta_2 + \frac{20}{9}\mathrm{H}_{0,0} + \frac{4}{3}\mathrm{H}_{0,0,0} \\
& - \frac{4}{3}\mathrm{H}_3 \Big] + (1-x) \Big[\frac{61}{9} - \frac{8}{3}\mathrm{H}_1 \Big] + (1+x) \Big[2\mathrm{H}_{0,0} - \frac{8}{3}\mathrm{H}_{-1,0} + \frac{41}{9}\mathrm{H}_0 - \frac{4}{3}\mathrm{H}_2 \Big] \Big) \\
& + 16 {C_F}^2 \Big(C_F - \frac{C_A}{2} \Big) \Big(p_{\rm qq}(-x) \Big[9\zeta_3 - 7\zeta_2^{\,2} + 12\mathrm{H}_{-3,0} - 64\mathrm{H}_{-2}\zeta_2 - 16\mathrm{H}_{-2,-1,0} - 6\mathrm{H}_{-2,0} \\
& + 52\mathrm{H}_{-2,0,0} + 56\mathrm{H}_{-2,2} - 12\mathrm{H}_{-1}\zeta_2 - 72\mathrm{H}_{-1}\zeta_3 - 16\mathrm{H}_{-1,-2,0} + 96\mathrm{H}_{-1,-1}\zeta_2 - 80\mathrm{H}_{-1,-1,0,0} \\
& - 96\mathrm{H}_{-1,-1,2} - 80\mathrm{H}_{-1,0}\zeta_2 - 6\mathrm{H}_{-1,0,0} + 44\mathrm{H}_{-1,0,0,0} + 12\mathrm{H}_{-1,2} + 8\mathrm{H}_{-1,2,0} + 64\mathrm{H}_{-1,3} + 3\mathrm{H}_0 \\
& + 3\mathrm{H}_0\zeta_2 + 26\mathrm{H}_0\zeta_3 + 28\mathrm{H}_{0,0}\zeta_2 + 9\mathrm{H}_{0,0,0} - 12\mathrm{H}_{0,0,0,0} - 12\mathrm{H}_2\zeta_2 - 6\mathrm{H}_3 - 4\mathrm{H}_{3,0} - 24\mathrm{H}_4 \Big] \\
& - (1-x) \Big[15 + 8\mathrm{H}_{-3,0} + 8\mathrm{H}_{-2,0,0} + 61\mathrm{H}_0 + 6\mathrm{H}_0\zeta_3 + 2\mathrm{H}_{0,0}\zeta_2 - 6\mathrm{H}_{0,0,0,0} + 12\mathrm{H}_1\zeta_2 + 60\mathrm{H}_1 \\
& + 8\mathrm{H}_{1,0} \Big] + (1+x) \Big[24\zeta_2 + 57\zeta_3 + 10\mathrm{H}_{-2,0} - 48\mathrm{H}_{-1}\zeta_2 - 4\mathrm{H}_{-1,0} + 40\mathrm{H}_{-1,0,0} + 48\mathrm{H}_{-1,2} \\
& + 59\mathrm{H}_0\zeta_2 - 22\mathrm{H}_{0,0} - 35\mathrm{H}_{0,0,0} - 22\mathrm{H}_2 - 4\mathrm{H}_{2,0} - 44\mathrm{H}_3 \Big] + 8\zeta_2 - 42\zeta_3 - 4\mathrm{H}_{-2,0} + 42\mathrm{H}_0 \\
& - 38\mathrm{H}_0\zeta_2 + 14\mathrm{H}_{0,0} - 16\mathrm{H}_2 + 26\mathrm{H}_{0,0,0} + 24\mathrm{H}_3 \Big) \; .
\end{aligned}
$$

$$
\begin{aligned}
P_{\rm ns}^{(2)s}(x) \; = \; & 16 n_f \frac{d^{abc}d_{abc}}{n_c} \Big(\frac{1}{2}(1-x) \Big[\frac{50}{3} + \frac{41}{12}\zeta_2 - \frac{5}{4}\zeta_2^{\,2} - \mathrm{H}_{-3,0} + \mathrm{H}_{-2}\zeta_2 - \mathrm{H}_{-2,0,0} + \frac{9}{4}\mathrm{H}_3 \\
& + 2\mathrm{H}_{-2,-1,0} + \frac{3}{2}\mathrm{H}_0\zeta_2 - \frac{1}{2}\mathrm{H}_1\zeta_2 - \frac{3}{4}\mathrm{H}_{1,0,0} + \frac{91}{12}\mathrm{H}_1 \Big] + \frac{1}{2}(1+x) \Big[\mathrm{H}_{-1,-1,0} - \frac{3}{2}\mathrm{H}_{-1}\zeta_2 + \frac{3}{4}\mathrm{H}_0 \\
& - \frac{13}{6}\mathrm{H}_{-1,0} + \frac{1}{2}\mathrm{H}_{-1,0,0} + 2\mathrm{H}_{-1,2} - \frac{3}{2}\mathrm{H}_{-2,0} + \frac{9}{4}\mathrm{H}_0\zeta_2 + \frac{29}{12}\mathrm{H}_{0,0} + \frac{41}{12}\mathrm{H}_2 - \mathrm{H}_2\zeta_2 - \frac{1}{2}\mathrm{H}_{2,0,0} \\
& + \frac{3}{2}\mathrm{H}_4 \Big] - \frac{1}{3}\Big(\frac{1}{x} + x^2 \Big) \Big[3\mathrm{H}_{-1}\zeta_2 + 2\mathrm{H}_{-1,-1,0} - 2\mathrm{H}_{-1,0,0} - 2\mathrm{H}_{-1,2} + \mathrm{H}_1\zeta_2 \Big] + \frac{1}{3}x^2 \Big[5\zeta_3 - 2\mathrm{H}_3 \\
& + 2\mathrm{H}_{-2,0} + 4\mathrm{H}_0\zeta_2 - 2\mathrm{H}_{0,0,0} + 2\mathrm{H}_1\zeta_2 \Big] + \frac{91}{24}\mathrm{H}_0 + \zeta_3 - \frac{9}{2}\zeta_2 + \zeta_2^{\,2} - \mathrm{H}_0\zeta_3 - \mathrm{H}_0\zeta_2 - 2\mathrm{H}_{0,0}\zeta_2 \\
& + \frac{3}{8}\mathrm{H}_{0,0} - \frac{1}{4}\mathrm{H}_{0,0,0} + \frac{1}{2}\mathrm{H}_{0,0,0,0} + \mathrm{H}_{-2,0} - \mathrm{H}_3 \Big) \; .
\end{aligned}
$$

NNLO singlet splitting functions

S.M., Vermaseren, Vogt ‘04



Coefficient functions in DIS at three loops

S.M., Vermaseren, Vogt ‘05

- Exact (analytical) results for coefficient functions of F_2 and F_L fill $\mathcal{O}(100)$ pages (normalsize fonts)

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Examples in particle physics

- Feynman diagram calculations
- QCD splitting functions
- ...