Two-dimensional harmonic polylogarithms for Bhabha scattering

Janusz Gluza

In collaboration with:

T. Riemann (DESY, Zeuthen)
M. Czakon (Würzburg U.)

26th May 2005

Outline

Outline

Methods of calculations for Feynman integrals in Particle Physics

- Numerical and analytical methods
- Differential equation method

Harmonic and Generalised Harmonic polylogarithms

- Definitions of HPLs
- Definitions of GPLs
- GPLs for the Bhabha process

Final remarks

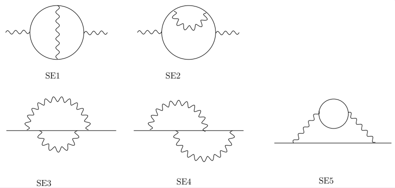
Outline

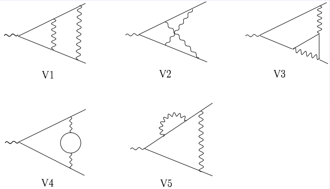
- Methods of calculations for Feynman integrals in Particle Physics
  - Numerical and analytical methods
  - Differential equation method
- Harmonic and Generalised Harmonic polylogarithms
  - Definitions of HPLs
  - Definitions of GPLs
  - GPLs for the Bhabha process
- Final remarks

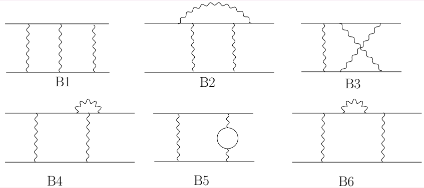
Outline

Bhabha scattering $e^+e^- \to e^+e^-$, $d = 4 - 2\epsilon$

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^dk_1 \ldots d^dk_L \;\; X}{(q_1^2 - m_1^2)^{\nu_1} \ldots (q_j^2 - m_j^2)^{\nu_j} (q_N^2 - m_N^2)^{\nu_N}},$$



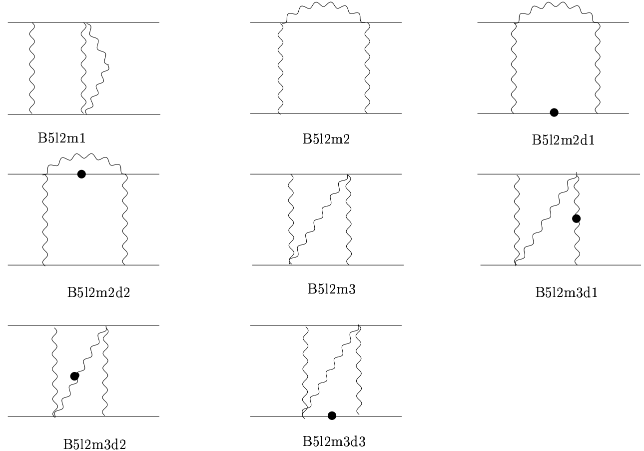




Bhabha scattering $e^+e^- \to e^+e^-$, Czakon, JG, Riemann, PRD2005

MI	B1	B2	B3	B4	B5	B6	Solved
B7l4m1	+	–	–	–	–	–	Y
B7l4m1N	+	–	–	–	–	–	Y
B7l4m2	–	+	–	–	–	–	Y/N
B7l4m2[d1--d3]	–	+	–	–	–	–	
B7l4m3	–	–	+	–	–	–	Y/N
B7l4m3[d1--d2]	–	–	+	–	–	–	
B6l3m1	+	–	+	–	–	–	
B6l3m1d	+	–	+	–	–	–	
B6l3m2	–	+	–	+	–	–	
B6l3m2d	–	+	–	+	–	–	
B6l3m3	–	–	+	–	–	–	
B6l3m3[d1--d5]	–	–	+	–	–	–	
B5l2m1	+	–	+	–	–	–	Y
B5l2m2	–	+	–	+	–	+	Y/N
B5l2m2[d1--d2]	–	+	–	+	–	+	Y/N
B5l2m3	+	–	+	–	–	–	
B5l2m3[d1--d3]	+	–	+	–	–	–	Y/N
B5l3m	–	+	+	+	–	–	
B5l3m[d1--d3]	–	+	+	+	–	–	
B5l4m	–	+	+	+	+	–	Y
B5l4md	–	+	+	+	+	–	Y
B4l2m*	–	–	–	+	+	+	Y

Bhabha scattering $e^+e^- \rightarrow e^+e^-$, Czakon, JG, Riemann, PRD2005



Numerical Methods

- No problem with the presence of different scales
- Problem: infrared singularities, thresholds and pseudothresholds

Solved:

- two-loop SE and vertex diagrams, multi-leg one-loop diagrams
  Ghinculov et al, 1995, *Passarino et al*, 2001
- Sector decomposition method: multiloop calculations in the non-physical region and evaluation of phase-space integrals
  Binoth-Heinrich, 2000
- SE by DEqs with Runge-Kutta method
  Caffo et al, 2002

Analytical Methods

- Problem: At most three different scales
- No problem with special regions

New methods:

- *Laporta-Remiddi* method: Master Integrals
- Expansion by regions
  V. Smirnov
- Mellin-Barnes transformation
  Davydychev, Smirnov
- Differential Equation Method
  Kotikov, Remiddi

Outline

General idea

M with respect to the external scale x,

$$\frac{d}{dx}M(x) = A(x)M(x) + B(x).$$

If $H(x)$ is a solution of the homogeneous equation

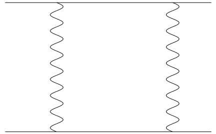
$$\frac{d}{dx}H(x) = A(x)H(x).$$

then the full solution is given by

$$M(x) = H(x)\left(\int^{x} H^{-1}(x')B(x')dx' + Const\right)$$

Solutions are generally combinations of [general] hypergeometric functions which are difficult to expand in powers of ϵ

1-loop box, Fleischer, Jegerlehner, Tarasov



$$\sim \frac{t-4m^2}{i\pi^{d/2}\Gamma\left(2-\frac{d}{2}\right)}\int \frac{d^d k_1}{k^2(k^2+2q_4k)(k+q_1+q_4)^2(k^2-2q_3k)}$$

$$= \cdots \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} F_1\left(\frac{d-3}{2},1,\frac{1}{2};\frac{d-1}{2};\frac{sZ(s,u,t)}{4},1-\frac{s}{4m^2}\right)$$

$$+\cdots F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};\frac{t}{t-4m^2},-m^2Z(s,u,t)\right)$$

$$+\cdots F^{1;2;1}_{1;1;0}\left[\begin{array}{llll}\frac{d-3}{2}: & \frac{d-3}{2},\ 1; & 1; & \\ \frac{d-1}{2}: & \quad\ \frac{d-2}{2}; & -; & \end{array} -m^2Z(s,u,t),1-\frac{4m^2}{s}\right]$$

ϵ expansion: Fleischer, Riemann, Tarasov, APP. **B34**, 5345

Appell hypergeometric functions:

$$F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}}\frac{\left(\frac{1}{2}\right)_s}{(1)_s}x^r y^s,$$

$$F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};x,y\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{3}{2}\right)_r\left(\frac{d-2}{2}\right)_s}x^r y^s$$

the Kampé de Fériet function:

$$F^{1;2;1}_{1;1;0}\left[\begin{matrix}\frac{d-3}{2}: & \frac{d-3}{2}, 1; & 1; \\ \frac{d-1}{2}: & \frac{d-2}{2}; & -;\end{matrix}\; x,y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}}\frac{\left(\frac{d-3}{2}\right)_r}{\left(\frac{d-2}{2}\right)_r}x^r y^s.$$

Order by order expansion in ϵ

$$M(x) = H(x)\left(\int^{x} H^{-1}(x')B(x')dx' + Const\right)$$

$$M = \sum_{i=-m}^{n} m^i \epsilon^i, \qquad A = \sum_{i=0}^{n+\alpha} a^i \epsilon^i, \qquad B = \sum_{i=-\alpha}^{n} b^i \epsilon^i$$

then

$$\boxed{\frac{d}{dx} m^i(x) = \sum_{j=0}^{m-i} a^j(x) m^{i-j}(x) + b^i(x)}$$

Conformal transformation

In physical situation, the differential operator is:

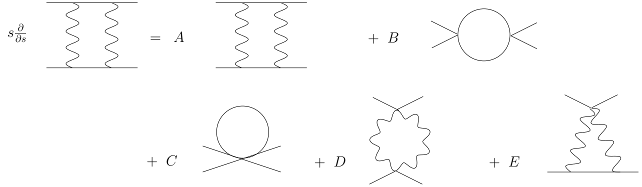
$$s\frac{\partial}{\partial s}=\frac{1}{2}\left\{\left(p_1^\mu+p_2^\mu\right)+\frac{s\left(p_2^\mu-p_3^\mu\right)}{s+t-4}\right\}\frac{\partial}{\partial p_2^\mu}.$$

Transformation

$$\begin{aligned} x &= \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}} \\ s &= -\frac{(1-x)^2}{x}, 0\leq x\leq 1 \end{aligned}$$

allows to change DEqs to the equations with factorized monomial denominators (!)

An example



e.g.:

$$A = \frac{8+s^2-2t+s(-6+t+\epsilon t)}{(-4+s)(-4+s+t)}$$

$$\longrightarrow \frac{1}{(1+x)^2}\frac{1}{(x+y)}\frac{1}{(1+xy)} \times \{\cdots\}$$

Functions which appear in calculations

Now we have:

$$
\begin{aligned}
\text{1LBox} &= \frac{2xy}{(1-x^2)(-1+y)^2}\left[\frac{1}{\epsilon}A + B + \epsilon C + \cdots\right] \\
A &= H[0,x] \\
B &= H[0,x](H[0,y] + 2H[1,y]) \\
C &= f\,\{\zeta_2, \zeta_3, \\
&\quad H[-1,x], H[0,x], H[-1,0,x], H[-1,0,0,x], ..., \\
&\quad H[0,y], H[1,y], H[0,0,0,y], ..., \\
&\quad G[-y,x], G[-1/y,0,x], G[-1/y,-1,0,x] \\
&\quad G[-1/y,0,0,x], G[-y,-1,0,x], G[-y,0,0,x], ..., \}
\end{aligned}
$$

If we can find solutions for ϵ^n, then it is easy to find solutions for $\epsilon^{(n+1)}$

Outline

Methods of calculations for Feynman integrals in Particle Physics
- Numerical and analytical methods
- Differential equation method

Harmonic and Generalised Harmonic polylogarithms
- Definitions of HPLs
- Definitions of GPLs
- GPLs for the Bhabha process

Final remarks

Vermaseren, Remiddi

$$\mathrm{H}(0;x) = \ln x \,,$$
$$\mathrm{H}(1;x) = \int_0^x \frac{dx'}{1-x'} = -\ln(1-x)\,,$$
$$\mathrm{H}(-1;x) = \int_0^x \frac{dx'}{1+x'} = \ln(1+x)$$

$$\mathrm{H}(\vec{0}_w;x) = \frac{1}{w!}\ln^w x$$
$$\mathrm{H}(\vec{m}_w;x) = \int_0^x dx'\, f(a;x')\,\mathrm{H}(\vec{m}_{w-1};x')$$

Bhabha process:

$$f(0,x) = \frac{1}{x},\quad f(1,x) = \frac{1}{1-x},\quad f(-1,x) = \frac{1}{1+x}$$

Vermaseren, Remiddi

$$
\begin{aligned}
H(0;x) &= \ln x \; , H(0,0;x) = \int \frac{H[0,x]}{x} dx = \frac{1}{2!} \ln^2 x , \ldots \\
H(1;x) &= \int_0^x \frac{dx'}{1-x'} = -\ln(1-x) \; , \\
H(-1;x) &= \int_0^x \frac{dx'}{1+x'} = \ln(1+x)
\end{aligned}
$$

$$
\begin{aligned}
H(0,1;x) &= \int_0^x \frac{dx'}{x'} H(1;x') = -\int_0^x \frac{dx'}{x'} \ln(1-x') \; , \\
H(0,1;x) &= \mathrm{Li}_2(x) \; , \\
H(1,0;x) &= \int_0^x \frac{dx'}{1-x'} H(0;x') = \int_0^x \frac{dx'}{1-x'} \ln x'
\end{aligned}
$$

Algebra, IBP identities

$$
\begin{aligned}
H(1,0;x) &= \int_0^x \frac{dx'}{1-x'} H(0;x') = \int_0^x \frac{dx'}{1-x'} \ln x' \,, \\
H(1,0;x) &= -\ln x \ln(1-x) + \mathrm{Li}_2(x) \\
H(1,0;x) &= H(0,x)H(1,x) - H(0,1;x)
\end{aligned}
$$

$$
\begin{aligned}
H(\vec{a},x)H(\vec{b},x) &= \sum_{\vec{c}=\vec{a}+\vec{b}} H(\vec{c},x) \\
H(m_1,...,m_q,x) &= H(m_1,x)H(m_2,...,m_q,x) \\
&\quad -H(m_1,m_2,x)H(m_3,...,m_q,x) \\
&\quad +... + (-1)^{q+1} H(m_q,...,m_1,x)
\end{aligned}
$$

Not all functions can be found analytically

M. Czakon, JG, T. Riemann, PRD71,073009: four not solved functions of weight 4 (see also Vermaseren, Remiddi):

$$
\begin{aligned}
H(-1,0,0,1,x) = I_1(x) &= \int_0^x \frac{dy}{1+y} \mathrm{Li}_3(y), \\
I_2(x) &= \int_0^x \frac{dy}{1+y} \mathrm{Li}_2(y) \ln(1-y), \\
I_3(x) &= \int_0^x \frac{dy}{1+y} \ln(y) \ln^2(1-y), \\
I_4(x) &= \int_0^x \frac{dy}{1+y} \ln^2(y) \ln(1-y).
\end{aligned}
$$

Additional basis functions

Bonciani, Aglietti: 2 and 3 massive propagators for two loop electroweak vertices

Thresholds: $x = 0, x = 1, x = 4,\ \frac{1}{1-x^2+1}$

$$f(\mp 4; x) = \frac{1}{4 \pm x}, \quad f(c(\overline{c}), x) = \frac{1}{x - \frac{1}{2} \mp i\frac{\sqrt{3}}{2}}$$

$$f(\mp r, x) = \frac{1}{\sqrt{x(4 \pm x)}}, \quad f(\mp 1 \mp r, x) = \frac{1}{\sqrt{x(4 \pm x)}\,(1 \pm x)},$$

$$f(\pm 1/4; x) = \frac{1}{\frac{1}{4} \mp x}, \quad f(\pm 1 \pm r/4; x) = \frac{1}{\sqrt{x \mp \frac{1}{4}}(1 \mp x)},$$

$$f(r_0/4; x) = \frac{1 - 2i\sqrt{x - \frac{1}{4}}}{x\sqrt{x - \frac{1}{4}}}, \quad f(-r_0/4; x) = \frac{1 - 2\sqrt{x + \frac{1}{4}}}{x\sqrt{x + \frac{1}{4}}}$$

Outline

$$g(-y,x) = \frac{1}{x+y}, \quad g(-1/y,x) = \frac{1}{x+\frac{1}{y}}$$

$$\boxed{G(\{-y,-1/y\},...,x) = \int_0^x dx' g(\{-y,-1/y\},x')H(...,x')}$$

e.g.:

$$\begin{aligned} G(-y;x) &= \int_0^x \frac{dx'}{x'+y} = \ln(1+\frac{x}{y})\,, \\ G(-1/y;x) &= \int_0^x \frac{dx'}{x'+\frac{1}{y}} = \ln(1+xy) \\ G(-y,0;x) &= \int_0^x \frac{dx'}{x'+y} H(0,x') \end{aligned}$$

Additional basis functions: Glover et al., massless two-loop diagrams with 3 off-shell legs

Kinematics:

$$\lambda = \sqrt{(1-x-y)^2-4xy} = \sqrt{(x-x_0)(x-x_1)}$$

$$x_0 = (1-\sqrt{y})^2, \qquad x_1 = (1+\sqrt{y})^2$$

$$f(\lambda,x) = \frac{1}{\lambda}, \quad f(x\lambda,x) = \frac{1}{x\lambda},$$

$$f(x_0,x) = -\frac{1}{x-x_0}, \quad f(x_1,x) = -\frac{1}{x-x_1}$$

$$e.g.: G(0;x,y) = \log\left(\frac{x}{x_0}\right),$$

$$G(\lambda;x,y) = \log\left(\frac{1-x+y-\lambda}{2\sqrt{y}}\right)$$

Outline

GPLs and HPLs to the weight 4, set={-1,0,1,-y,-1/y}

weight 1, #5	weight 2, #25	weight 3, #125	weight 4, #125
2 GPLs 3 HPLs	16 GPLs 9 HPLs	98 GPLs 27 HPLs	544 GPLs 81 HPLs

```
(*
 weight 2, 16 GPLs, 9 HPLs  = 25

 irreducible: G[-y_, x],
              G[-y_,-1,x_],
              G[-y_,0,x_],
              G[-y_,1,x_],
              G[-y_,-1/y,x_]
*)

G[-y_,-1,x_]:= (Log[1 + x]*Log[(x + y)/(-1 + y)] -
                PolyLog[2, (1 - y)^(-1)] +
                PolyLog[2, (1 + x)/(1 - y)] )

G[-1/y_,-1,x_]:= G[-y,-1,x] /. y -> 1/y

G[-y_,-y_,x_]:= 1/2*G[-y,x]^2

...

G[-y_, 1, 1,x_]:= ((Log[1 - x]^2*Log[(x + y)/(1 + y)])/2 +
                    Log[1 - x]*PolyLog[2, (1 - x)/(1 + y)] +
                    PolyLog[3, (1 + y)^(-1)] -
                    PolyLog[3, (1 - x)/(1 + y)])


G[-y_,-(y_),-1,x_] := NIntegrate[G[-y,-1,z]/(y+z),{z,0,x}];

...
G[-(y_)^(-1),-(y_)^(-1),-1,-1,x_] :=
       NIntegrate[ If[z2 < z1, G[-1,-1,z2]v^2/((1 + v*z1)*(1 + v*z2))
                     /.v -> y, 0], {z1,0,x},{z2,0,z1}];
...
```

```
(*
 weight 2, 16 GPLs, 9 HPLs  = 25

 irreducible: G[-y_, x],
              G[-y_,-1,x_],
              G[-y_,0,x_],
              G[-y_,1,x_],
              G[-y_,-1/y,x_]
*)

G[-y_,-1,x_]:= (Log[1 + x]*Log[(x + y)/(-1 + y)] -
                PolyLog[2, (1 - y)^(-1)] +
                PolyLog[2, (1 + x)/(1 - y)] )

G[-1/y_,-1,x_]:= G[-y,-1,x] /. y -> 1/y

G[-y_,-y_,x_]:= 1/2*G[-y,x]^2

...

G[-y_, 1, 1,x_]:= ((Log[1 - x]^2*Log[(x + y)/(1 + y)])/2 +
                    Log[1 - x]*PolyLog[2, (1 - x)/(1 + y)] +
                    PolyLog[3, (1 + y)^(-1)] -
                    PolyLog[3, (1 - x)/(1 + y)])


G[-y_,-(y_),-1,x_] := NIntegrate[G[-y,-1,z]/(y+z),{z,0,x}];

...
G[-(y_)^(-1),-(y_)^(-1),-1,-1,x_] :=
       NIntegrate[ If[z2 < z1, G[-1,-1,z2]v^2/((1 + v*z1)*(1 + v*z2))
                             /.v -> y, 0], {z1,0,x},{z2,0,z1}];
...
```

- Relations between GPLs in s and t channels

$$G(-y,-1,x) \longrightarrow H(-1,x)H(1,y) - H(0,x)H(1,y) + H(-1,x)H(-x,y) + H(0,-1,x) + H(1,0,y) - G(1,-x,y)$$

- Two different transformations

$$x = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}} \quad and \quad z = \frac{1}{\sqrt{1-4/s}}$$

then

$$x = \frac{1-z}{1+z}$$

- $$x = \frac{1-x'}{1+x'}, \quad y = \frac{1-y'}{1+y'}$$
$$H\left(-1,-1,\frac{1-x}{1+x}\right) \longrightarrow H(-1,-1,x') - H(-1,x')H(-1,1) + H(-1,-1,1)$$
$$G(-y,0,x) \longrightarrow H\left(-\frac{1-y'}{1+y'},0,1\right) + \int_1^x dz \frac{H(0,z)}{x' + \frac{1-y'}{1+y'}}$$
$$= \{-\zeta_2 + H(0,-1,y') - H(0,0,y')\}$$
$$+ \int_0^{x'} dz[H(1,z) + H(-1,z)] \times \left[\frac{1}{1+z} + \frac{y'}{1-zy'}\right]$$
$$= \{...\} + ... + H(-1,1,x') + G(1/y',-1,x) + G(1/y',1,x)$$
- Formally, generating functions with indices {-1,0,1,-y,-1/y} must be extended by generating functions with indices +y, +1/y

- Calculations of Feynman integrals with DEqs method: new functions
- results can be represented analyticaly in an easy way in order by order ϵ parameter expansion
- Mathematica codes for algebraic manipulations [$x \leftrightarrow y$, conformal transformations] and numerical calculation of HPLs and GPLs are in use for Bhabha calculations (Fortran code by Gehrmann and Remiddi for numerical calculation of HPLs and GPLs exists either)