On Computation of Gröbner Bases for Linear Difference Systems

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Generation of Difference Schemes

Consider PDEs in the conservation law form

$$\frac{\partial \mathbf{v}}{\partial x} + \frac{\partial}{\partial y}\mathbf{F}(\mathbf{v}) = 0 \Longleftrightarrow \oint_{\Gamma} -\mathbf{F}(\mathbf{v})dx + \mathbf{v}dy = 0\,.$$

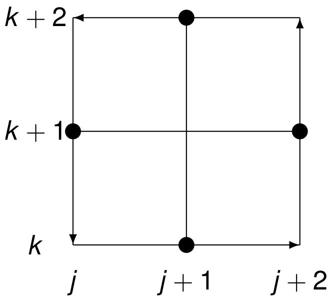
Γ is arbitrary closed contour, $\mathbf{v}$ is a $m-$vector function in unknown $n-$vector function $\mathbf{u}$ and its partial derivatives $\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_{xx}, \mathbf{u}_{xy}, \mathbf{u}_{yy}, \ldots$
F is a function that maps R^m into R^m.

To do discretization we set

$$\mathbf{u}(x,y) = \mathbf{u}(x_j, y_k) \equiv \mathbf{u}_{j\,k},\ \mathbf{u}_x(x,y) = \mathbf{u}_x(x_j, y_k) \equiv (\mathbf{u}_x)_{j\,k}, \ldots$$

Generation of Difference Schemes

Choose the integration contour and add the integral relations, e.g.,



$$\int_{x_j}^{x_{j+2}} \mathbf{u}_x dx = \mathbf{u}(x_{j+2}, y) - \mathbf{u}(x_j, y), \quad \int_{y_k}^{y_{k+2}} \mathbf{u}_y dy = \mathbf{u}(x, y_{k+2}) - \mathbf{u}(x, y_k), \ldots .$$

Generation of Difference Schemes

Using a numerical integration method, e.g. the midpoint one, with

$$x_{j+1} - x_j = y_{k+1} - y_k = \triangle h$$

we rewrite the equations and the relations as

$$\begin{gathered}
-(\mathbf{F}(\mathbf{v})_{j+1\,k} - \mathbf{F}(\mathbf{v})_{j+1\,k+2}) + (\mathbf{v}_{j+2\,k+1} - \mathbf{v}_{j\,k+1}) = 0, \\
(\mathbf{u}_x)_{j+1\,k} \cdot 2\triangle h = \mathbf{u}_{j+2\,k} - \mathbf{u}_{j\,k}, \\
(\mathbf{u}_y)_{j\,k+1} \cdot 2\triangle h = \mathbf{u}_{j\,k+2} - \mathbf{u}_{j\,k}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered}$$

A difference scheme for $\mathbf{u}$ is obtained (Mozzhilkin, Blinkov'01) by elimination of all partial derivatives $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_{xx}$, ... from the above system. The elimination can be achieved by constructing a Gröbner basis (GB), if it exists (finite). For linear PDEs GB always exists.

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Reduction of Feynman Integrals

Consider scalar $L-$loop integral with n internal lines

$$\mathcal{I}_\nu := \int d^d k_1 \cdots d^d k_L \, \frac{1}{\prod_{i=1}^{n} P_i^{\nu_i}} \, .$$

P_i are propagators and $\nu = \{\nu_1, \nu_2, \ldots, \nu_n\} \in \mathbb{Z}^n$ is multi-index.

$\mathcal{I}_\nu$ satisfies recurrence relations (RR) derived from the integration by part method (Chetyrkin, Tkachov'81).

After a proper shift of indices $\mu = \nu - \lambda$, $\lambda \in \mathbb{Z}^n_{\geq 0}$, RR can be written in the form

$$f_j := \sum_\alpha b^j_\alpha \, \theta^\alpha \circ \mathcal{I}_\mu = 0 \, , \qquad j = 1, \ldots, p \, .$$

Reduction of Feynman Integrals

$\theta^\alpha = \theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n}$, $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{Z}^n_{\geq 0}$. θ_i denotes the right-shift operator for the i-th index, i.e.,

$$\theta_i \circ \mathcal{I}_\mu = \mathcal{I}_{\mu_1,\ldots,\mu_i+1,\ldots,\mu_n} \, .$$

Coefficients b^j_α are polynomials in indices $\{\nu_1, \ldots, \nu_n\}$ and physical parameters: masses, scalar products of external momenta, space-time dimension d.

Converting difference polynomials f_j into the Gröbner basis form allows (Gerdt'04):

- Define basic (master) integrals as those independent modulo RR.
- Reduce an integral $\mathcal{I}_{\bar\nu}$ with shifted indices $\nu \longrightarrow \bar\nu$ to the basic integrals by using the standard Gröbner reductions.

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Difference Algebra

Let $\{y^1, \ldots, y^m\}$ be the set of *indeterminates* such, for example, as functions of $n-$variables $\{x_1, \ldots, x_n\}$ and $\theta_1, \ldots, \theta_n$ be the set of mutually commuting *difference operators (differences)*, e.g.,

$$\theta_i \circ y^j = y^j(x_1, \ldots, x_i + 1, \ldots, x_n).$$

A *difference ring R with differences* $\theta_1, \ldots, \theta_n$ is a commutative ring R such that $\forall f, g \in R,\ 1 \leq i, j \leq n$

$$\theta_i\theta_j = \theta_j\theta_i,\ \theta_i \circ (f + g) = \theta_i \circ f + \theta_i \circ g,\ \theta_i \circ (f\,g) = (\theta_i \circ f)(\theta_i \circ g)$$

Similarly one defines a *difference field*.

Difference Algebra

Let $\mathbb{K}$ be a difference field. Denote by $\mathbb{R} := \mathbb{K}\{y^1, \ldots, y^m\}$ the difference ring of polynomials over $\mathbb{K}$ in variables

$$\{ \theta^\mu \circ y^k \mid \mu \in \mathbb{Z}^n_{\geq 0},\, k = 1, \ldots, m \}.$$

Denote by $\mathbb{R}_L$ the set of linear polynomials in $\mathbb{R}$ and use the notations

$$\Theta = \{ \theta^\mu \mid \mu \in \mathbb{Z}^n_{\geq 0} \},\ \deg_i(\theta^\mu \circ y^k) = \mu_i,\ \deg(\theta^\mu \circ y^k) = |\mu| = \sum_{i=1}^{n} \mu_i\,.$$

A *difference ideal* I in $\mathbb{R}$ is an ideal $I \in \mathbb{R}$ close under the action of any operator from Θ. If $F := \{f_1, \ldots, f_k\} \subset \mathbb{R}$ is a finite set, then the smallest difference ideal containing F denoted by $\mathrm{Id}(F)$. If $F \subset \mathbb{R}_L$, then $\mathrm{Id}(F)$ is *linear difference ideal*.

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Difference Algebra

A total ordering $\prec$ over the set of $\theta_\mu y^j$ is a *ranking* if it satisfies

1. $\theta_i \theta^\mu \circ y^j \succ \theta^\mu \circ y^j$
2. $\theta^\mu y^j \succ \theta^\nu \circ y^k \quad \Longleftrightarrow \quad \theta_i \theta^\mu \circ y^j \succ \theta_i \theta^\nu \circ y^k \quad \forall\, i, j, k, \mu, \nu.$

If $\mu \succ \nu \Longrightarrow \theta_\mu \circ y^j \succ \theta_\nu \circ y^k$ the ranking is *orderly*.
If $i \succ j \Longrightarrow \theta_\mu \circ y^j \succ \theta_\nu \circ y^k$ the ranking is *elimination*.

Given a ranking $\succ$, every linear polynomial $f \in \mathbb{R}_L \setminus \{0\}$ has the *leading term* $a\,\theta \circ y^j$, $\theta \in \Theta$; $\mathrm{lc}(f) := a \in \mathbb{K} \setminus \{0\}$ is the *leading coefficient* and $\mathrm{lm}(f) := \theta \circ y^j$ is the *leading monomial*.
In $\mathbb{R}_L$ a ranking is a *monomial order*. If $F \in \mathbb{R}_L$, $\mathrm{lm}(F)$ is the set of the leading monomials and $\mathrm{lm}_j(F)$ is its subset with indeterminate y^j. Thus,

$$\mathrm{lm}(F) = \cup_{j=1}^{m} \mathrm{lm}_j(F)\,.$$

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Gröbner Bases

Given nonzero linear difference ideal $I = \mathrm{Id}(G)$ and term order $\succ$, its generating set $G = \{g_1, \dots, g_s\} \subset \mathbb{R}_L$ is a *Gröbner basis* (GB) (Buchberger, Winkler'98, Mikhalev et al'99) of I if

$$\forall f \in I \cap \mathbb{R}_L \setminus \{0\}\ \exists g \in G, \theta \in \Theta\ :\ \mathrm{lm}(f) = \theta \circ \mathrm{lm}(g)\,.$$

It follows that $f \in I$ *is reducible modulo G*

$$f \xrightarrow[g]{} f' := f - \mathrm{lc}(f)\,\theta \circ (g/lc(g)), \quad f' \in I, \dots, \quad f \xrightarrow[G]{} 0\,.$$

Similarly, a polynomial $h \in \mathbb{R}_L$, whose terms are reducible (if any) modulo set $F \in \mathbb{R}_L$, can be reduced to an irreducible polynomial $\bar{h}$, which is said to be in the *normal form modulo F* ($\bar{h} = NF(h, F)$).

Gröbner Bases

In our algorithmic construction of GB we shall use a restricted set of reductions called *Janet-like* (Gerdt, Blinkov'05) and defined as follows.

For a finite set $F \in \mathbb{R}_L$ and order $\succ$, partition every $\mathrm{lm}_k(F)$ groups labeled by $d_0, \ldots, d_i \in \mathbb{Z}_{\geq 0}$, $(0 \leq i \leq n)$, $\;([0]_k = \mathrm{lm}_k(F))$

$$[d_0, \ldots, d_i]_k := \{u \in \mathrm{lm}_k(F) \mid d_0 = 0, d_1 = \deg_1(u), ..., d_i = \deg_i(u)\}.$$

Define $h_i(u, \mathrm{lm}_k(F)) := \max\{\deg_i(v) \mid u, v \in [d_0, ..., d_{i-1}]_k\} - \deg_i(u)$.
If $h_i(u, \mathrm{lm}_k(F)) > 0$, then $\theta_i^{s_i}$ where

$$s_i := \min\{\deg_i(v) - \deg_i(u) \mid u, v \in [d_0, ..., d_{i-1}]_k, \deg_i(v) > \deg_i(u)\}$$

is called a *difference power* for u.

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Gröbner Bases

Denote the set of difference powers for $u \in \mathrm{lm}_k(F)$ by $DP(u, \mathrm{lm}(F))$ and define the following subset of Θ

$$\mathcal{J}(u, \mathrm{lm}(F)) := \{\theta \in \Theta \mid \forall \vartheta_i^{s_i} \in DP(u, \mathrm{lm}(F)) \ : \ \deg_i(\theta \circ u) < s_i\}.$$

A GB of $I = \mathrm{Id}(G)$ is called Janet-like (Gerdt, Blinkov'05) if

$$\forall f \in I \cap \mathbb{R}_L \setminus \{0\} \ \exists g \in G, \theta \in \mathcal{J}(\mathrm{lm}(g), \mathrm{lm}(G)) : \ \mathrm{lm}(f) = \theta \circ \mathrm{lm}(g)\,.$$

This implies $\mathcal{J}-$reductions and $\mathcal{J}-$normal form: $NF_{\mathcal{J}}(f, F)$.

Algorithmic characterization of Janet-like GB:

$$\forall g \in G \ \forall \vartheta \in DP(\mathrm{lm}(g), \mathrm{lm}(G)) : \ NF_{\mathcal{J}}(\vartheta \circ g, G) = 0\,.$$

They are similar to (but more compact than) involutive Janet bases.

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Algorithm: Janet-like Gröbner Basis($F \subset \mathbb{R}_L, \succ$)

choose $f \in F$ with the lowest $\mathrm{lm}(f)$ w.r.t. $\succ$
$G := \{f\};\ \ Q := F \setminus G$
do
    $h := 0$
    while $Q \neq \emptyset$ and $h = 0$ **do**
        choose $p \in Q$ with the lowest $\mathrm{lm}(p)$ w.r.t. $\succ$
        $Q := Q \setminus \{p\};\ \ h :=$ **Normal Form**$(p, G, \prec)$
    od
    if $h \neq 0$ **then**
        for all $\{g \in G \mid \mathrm{lm}(g) = \theta^{\mu}(\mathrm{lm}(h)), |\mu| > 0\}$ **do**
            $Q := Q \cup \{g\};\ \ G := G \setminus \{g\}$
        od
        $G := G \cup \{h\}$
        $Q := Q \cup \{\, \theta^{\beta} \circ g \mid g \in G,\ \beta \in DP(\mathrm{lm}(g), \mathrm{lm}(G))\,\}$
    fi
od while $Q \neq \emptyset$
return G

Subalgorithm

Algorithm: Normal Form$(p, G, \prec)$

1: $h := p$
2: **while** $h \neq 0$ **and** h has a monomial u with coefficient $b \in \mathbb{K}$ $\mathcal{J}-$reducible modulo G **do**
3: **take** $g \in G$ s.t. $u = \theta^{\gamma}(\mathrm{lm}(g))$ with $\gamma \in \mathcal{J}(\mathrm{lm}(g), \mathrm{lm}(G))$
4: $h := h/b - \theta^{\gamma} \circ (g/\mathrm{lc}(g))$
5: **od**
6: **return** h

Algorithm **Janet-like Gröbner Basis** implemented (in an improved form) in Maple (Gerdt, Robertz'05) is an extension of the polynomial algorithm (Gerdt, Blinkov'05) to difference ideals.

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FDS for Laplace Equation

Consider the Laplace equation $u_{xx} + u_{yy} = 0$ and rewrite it as the conservation law

$$\oint_{\Gamma} -u_y dx + u_x dy = 0\,.$$

Add the integral relations

$$\int_{x_j}^{x_{j+2}} u_x dx = u(x_{j+2}, y) - u(x_j, y), \quad \int_{y_k}^{y_{k+2}} u_y dy = u(x, y_{k+2}) - u(x, y_k)\,.$$

Thus, we obtain 3 integral relations for 3 unknown functions

$$u(x,y),\ u_x(x,y),\ u_y(x,y)\,.$$

FDS for Laplace Equation

Choose midpoint integration method for above rectangular contour.

This yields the discrete system

$$\begin{cases} -((u_y)_{j+1\,k} - (u_y)_{j+1\,k+2}) + ((u_x)_{j+2\,k+1} - (u_y)_{j\,k+1}) = 0, \\ (u_x)_{j+1\,k} \cdot 2\triangle h = u_{j+2\,k} - u_{j\,k}, \\ (u_y)_{j\,k+1} \cdot 2\triangle h = u_{j\,k+2} - u_{j\,k}. \end{cases}$$

Its difference form is

$$\begin{cases} (\theta_x\theta_y^2 - \theta_x) \circ u_y + (\theta_x^2\theta_y - \theta_y) \circ u_x = 0\,, \\ 2\triangle h\,\theta_x \circ u_x - (\theta_x^2 - 1) \circ u = 0\,, \\ 2\triangle h\,\theta_y \circ u_y - (\theta_y^2 - 1) \circ u = 0\,. \end{cases}$$

FDS for Laplace Equation

Computation of GB (in this case Janet-like GB is the reduced GB) for elimination order with $u_x \succ u_y \succ u$ and $\theta_x \succ \theta_y$ gives

$$\begin{cases} \theta_x \circ u_x - \frac{1}{2\triangle h}\,(\theta_x^2 - 1) \circ u = 0\,, \\ \theta_y \circ u_x + \theta_x \circ u_y - \frac{1}{2\triangle h}\,(\theta_x\theta_y((\theta_x^2 - 1) + (\theta_y^2 - 1))) \circ u = 0\,, \\ \theta_x^2 \circ u_y - \frac{1}{2\triangle h}\,(\theta_x^2\theta_y((\theta_x^2 - 1) + (\theta_y^2 - 1)) - \theta_y(\theta_x^2 - 1)) \circ u = 0\,, \\ \theta_y \circ u_y - \frac{1}{2\triangle h}\,(\theta_y^2 - 1) \circ u = 0\,, \\ \frac{1}{2\triangle h}\,(\theta_x^4\theta_y^2 + \theta_x^2\theta_y^4 - 4\theta_x^2\theta_y^2 + \theta_x^2 + \theta_y^2) \circ u = 0\,. \end{cases}$$

FDS for Laplace Equation

The last equation gives the difference scheme written in double nodes

$$\frac{u_{j+2\,k} - 2u_{j\,k} + u_{j-2\,k}}{4\triangle h^2} + \frac{u_{j\,k+2} - 2u_{j\,k} + u_{j\,k-2}}{4\triangle h^2} = 0.$$

Similarly, the trapezoidal rule for the relation integrals generates the same difference scheme but written in ordinary nodes

$$\frac{u_{j+1\,k} - 2u_{j\,k} + u_{j-1\,k}}{\triangle h^2} + \frac{u_{j\,k+1} - 2u_{j\,k} + u_{j\,k-1}}{\triangle h^2} = 0.$$

Conclusions

- GB are the most universal algorithmic tool for linear difference systems.
- In particular, they can be applied to generate differences schemes for linear PDEs and to reduce multiloop Feynman integrals.
- There is an efficient algorithm for construction of GB for linear difference ideals. The algorithm is based on the concept of Janet-like reductions.
- Janet-like GB are similar to (but more compact than) involutive Janet bases, and the reduced GB can be easily extracted from the Janet-like GB without any extra computational costs.
- The first implementation in Maple is already available.
- Computer experiments and open software for constructing polynomial Janet and Janet-like bases presented on the Web site `http://invo.jinr.ru`.

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