Geometrical Methods in Loop Calculations

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Earlier Papers: Singularities, Reduction, etc.

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G. Källén and A. Wightman, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, No.6 (1958) 1

S. Mandelstam, Phys. Rev. **115** (1959) 1742

R.E. Cutkosky, J. Math. Phys. **1** (1960) 429

J.C. Taylor, Phys. Rev. **117** (1960) 261

Yu.A. Simonov, Sov. Phys. JETP **16** (1963) 1599

R. Karplus, C.M. Sommerfield and E.H. Wichmann, Phys. Rev. **114** (1959) 376

A.C.T. Wu, Mat. Fys. Medd. Dan. Vid. Selsk. **33**, No.3 (1961) 1

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G. Källén and J. Toll, J. Math. Phys. **6** (1965) 299 (1965)

B.G. Nickel, J. Math. Phys. **19** (1978) 542

N. Ortner and P. Wagner, Ann. Inst. Henri Poincaré (Phys. Théor.) **63** (1995) 81

P. Wagner, Indag. Math. **7** (1996) 527

Dimensional Regularization

One of the most powerful tools used in loop calculations is dimensional regularization.

G. 'tHooft and M. Veltman, Nucl. Phys. **B44** (1972) 189;

C.G. Bollini and J.J. Giambiagi, Nuovo Cimento **12B** (1972) 20;

J.F. Ashmore, Lett. Nuovo Cim. **4** (1972) 289;

G.M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972) 329.

In some cases, one can derive results valid for an arbitrary space-time dimension $n = 4 - 2\varepsilon$, usually in terms of various hypergeometric functions. For applications, explicit results for the terms of the expansion in ε are needed.

Geometrical Approach

To predict types of functions (and values of their arguments) which may appear in higher orders of ε-expansion, a geometrical approach happens to be very useful. It is summarized in the paper

A.I. Davydychev and R. Delbourgo, J. Math. Phys. **39** (1998) 4299.

Using this approach, the results for *all* terms of the ε-expansion have been obtained for the one-loop two-point function with arbitrary masses, one-loop three-point integrals with massless internal lines and arbitary (off-shell) external momenta and two-loop vacuum diagrams with arbitrary masses.

A.I. Davydychev, Phys. Rev. **D61** (2000) 087701;

A.I. Davydychev and M.Yu. Kalmykov, Nucl. Phys. B (PS) **89** (2000) 283; Nucl. Phys. **B605** (2001) 266

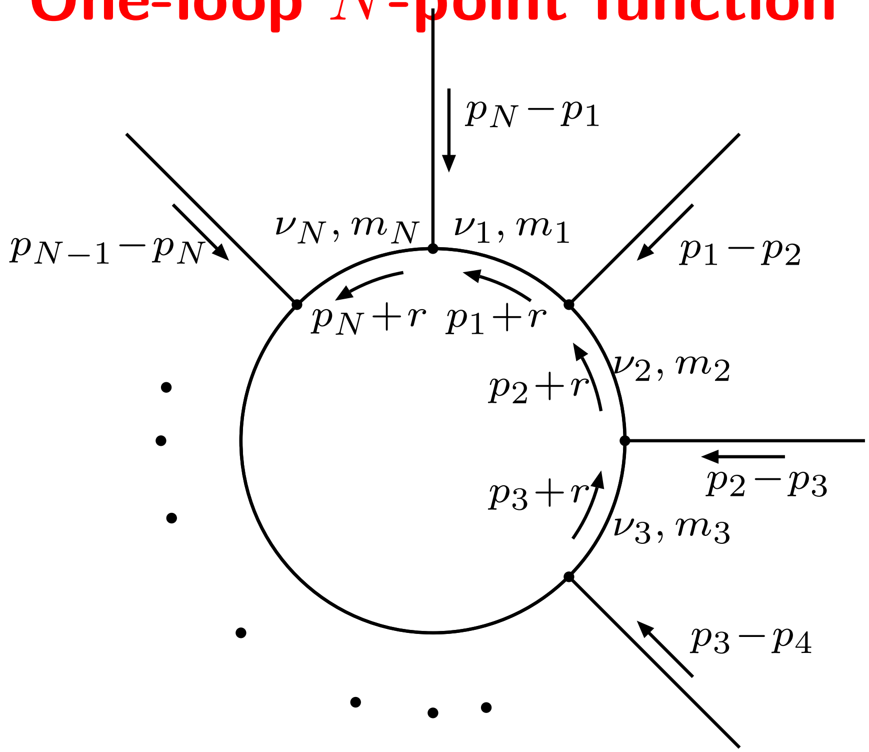
These results have been represented in terms of

$$\mathsf{Ls}_j(\theta) = -\int_0^\theta \mathrm{d}\phi \; \ln^{j-1}\left|2\sin\frac{\phi}{2}\right| ,$$

whose angular arguments have a rather transparent geometrical interpretation (angles of certain triangles).

In more complicated cases, generalizations of log-sine integrals (or more complex functions of angular variables) appear.

One-loop N-point function



$$k_{jl}^2 = (p_j - p_l)^2$$

Feynman parameters:

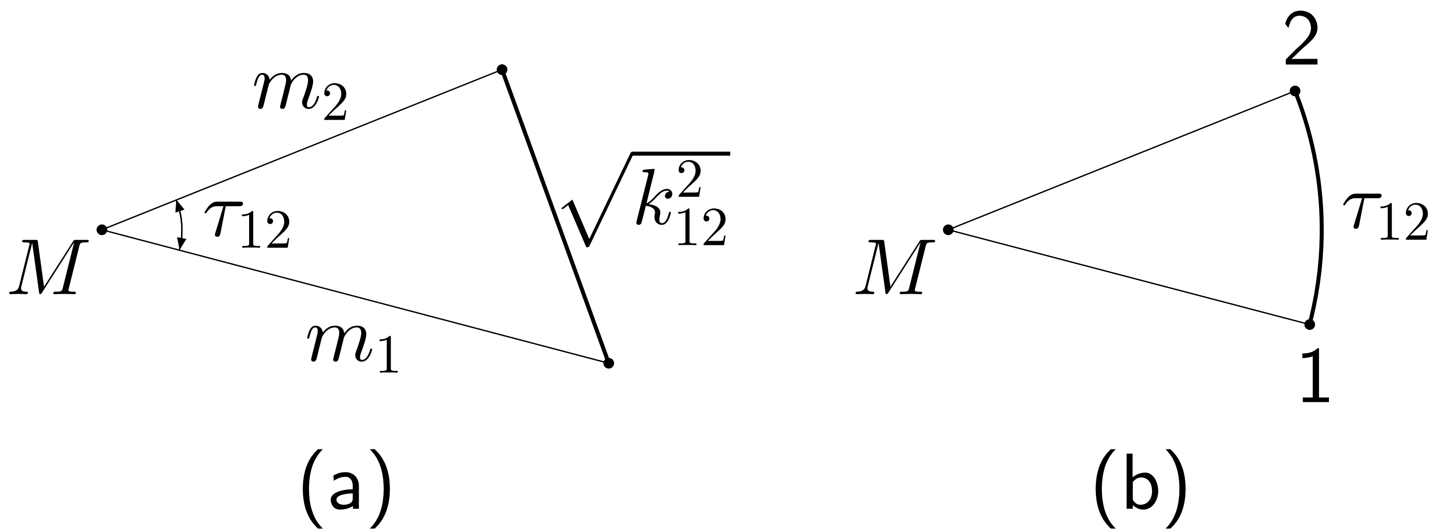
$$J^{(N)}\left(n;\nu_1,\ldots,\nu_N\right) = \mathsf{i}^{1-n}\pi^{n/2}\,\frac{\Gamma\left(\sum\nu_i - n/2\right)}{\prod\Gamma\left(\nu_i\right)}$$

$$\times \int\limits_0^1 \ldots \int\limits_0^1 \frac{\prod \alpha_i^{\nu_i-1}\mathsf{d}\alpha_i\;\delta\left(\sum\alpha_i - 1\right)}{\left[\sum\sum\limits_{j<l}\;\alpha_j\alpha_l k_{jl}^2 - \sum\alpha_i m_i^2\right]^{\Sigma\nu_i - n/2}}$$

It can be transformed to an integral over the interior of the N-dimensional solid angle $\Omega^{(N)}$ of the basic simplex,

$$J^{(N)}(n;1,\ldots,1) = 2\mathsf{i}^{1-2N}\pi^{n/2}\,\Gamma\left(N-\tfrac{n}{2}\right)\,\frac{m_0^{n-N-1}}{\sqrt{\Lambda^{(N)}}}\;\int\limits_{\Omega^{(N)}}\!\!\ldots\!\int\;\frac{\prod\mathsf{d}\gamma_i}{\gamma_N^{n-N}}\,\delta\left(\sum\gamma_i^2 - 1\right)$$

General two-point function, geometrical approach



Two-point case: (a) the basic triangle and (b) the arc τ_{12}

$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}$$

General two-point function, geometrical approach

$$J^{(2)}(4-2\varepsilon;1,1) = \mathrm{i}\pi^{2-\varepsilon}\Gamma(\varepsilon)\,\frac{m_0^{1-2\varepsilon}}{\sqrt{k_{12}^2}}\left\{\Omega_1^{(2;4-2\varepsilon)} + \Omega_2^{(2;4-2\varepsilon)}\right\}$$

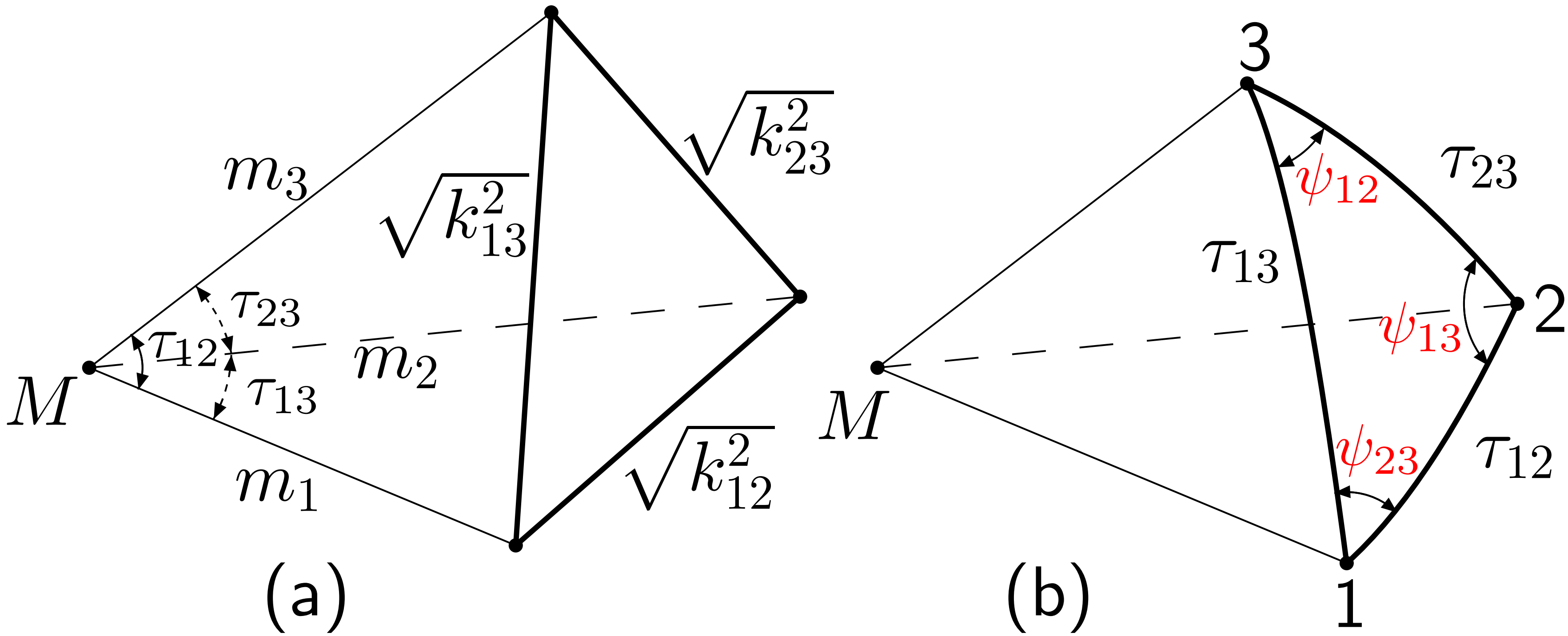
with

$$\Omega_i^{(2;4-2\varepsilon)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos^{2-2\varepsilon}\theta} = \tan\tau_{0i}\;{}_2F_1\left(\begin{array}{c} 1/2, \varepsilon \\ 3/2 \end{array}\middle| -\tan^2\tau_{0i}\right)$$

$$c_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \quad D^{(2)} = 1-c_{12}^2 = \sin^2\tau_{12}, \quad m_0 = m_1m_2\sqrt{\frac{D^{(2)}}{k_{12}^2}},$$

$$\cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01}+\tau_{02} = \tau_{12}.$$

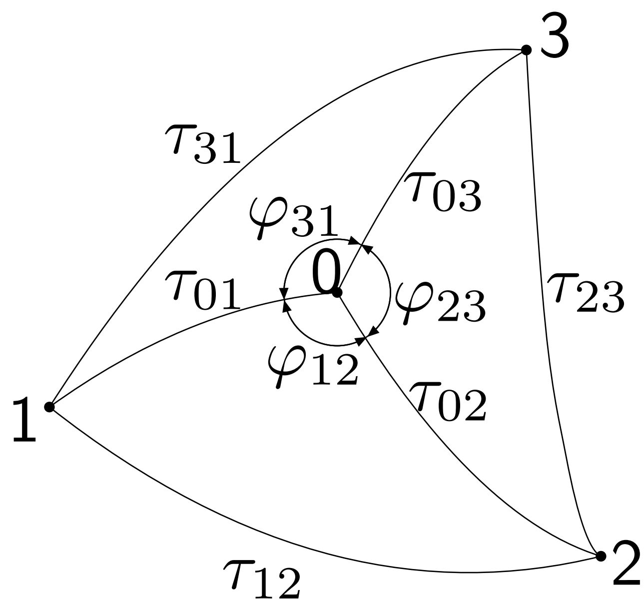
General three-point function: geometrical approach



Three-point case: (a) the basic tetrahedron and (b) the solid angle

General three-point function: geometrical approach

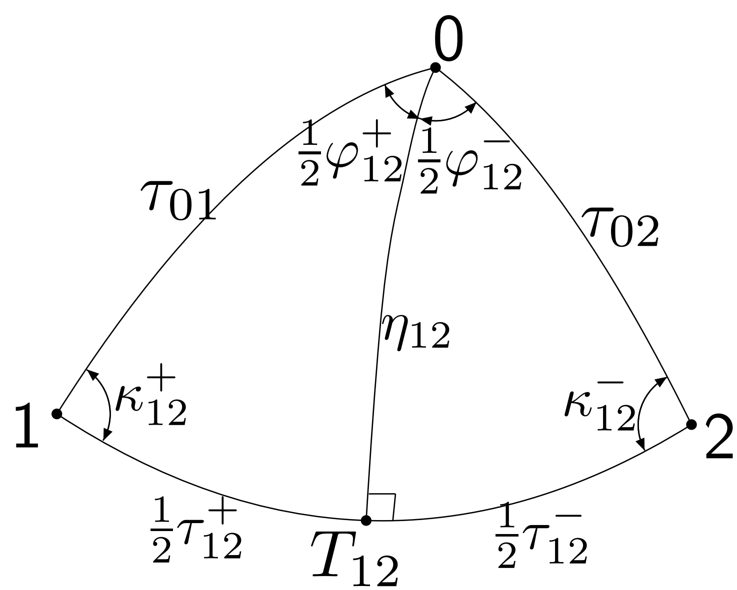
Relation to the angles accociated with a spherical (or hyperbolic) triangle:



$$\varphi_{12} + \varphi_{23} + \varphi_{31} = 2\pi, \quad \cos\tau_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}, \quad \text{etc.}; \quad \cos\tau_{0i} = \frac{m_0}{m_i},$$

$$m_0 = m_1 m_2 m_3 \sqrt{D^{(3)}/\Lambda^{(3)}}, \quad \Lambda^{(3)} = \tfrac{1}{4}\Delta(k_{12}^2, k_{23}^2, k_{31}^2), \quad D^{(3)} = \det \| \cos\tau_{jl} \| \quad (j, l = 1, 2, 3)$$

One of the three triangles ($\frac{1}{2}(\varphi_{12}^{+} + \varphi_{12}^{-}) = \varphi_{12}$):



Three-point function in $n = 4 - 2\varepsilon$ dimensions:

$$J^{(3)}(n;1,1,1) = -\frac{\mathrm{i}\pi^{n/2}}{\sqrt{\Lambda^{(3)}}}\,\Gamma\left(3-\frac{n}{2}\right)\; m_0^{n-4}\,\Omega^{(3;n)}\;,$$

with

$$\Omega^{(3;n)} = \int\!\!\int\limits_{\Omega^{(3)}} \frac{\sin^{n-2}\theta\;\mathrm{d}\theta\;\mathrm{d}\phi}{\cos^{n-3}\theta}\;.$$

According to a geometrical approach,

$$\begin{aligned}\Omega^{(3;n)} \quad = \quad & \omega\left(\tfrac{1}{2}\varphi_{12}^{+},\eta_{12}\right)+\omega\left(\tfrac{1}{2}\varphi_{12}^{-},\eta_{12}\right)\\ & +\omega\left(\tfrac{1}{2}\varphi_{23}^{+},\eta_{23}\right)+\omega\left(\tfrac{1}{2}\varphi_{23}^{-},\eta_{23}\right)\\ & +\omega\left(\tfrac{1}{2}\varphi_{31}^{+},\eta_{31}\right)+\omega\left(\tfrac{1}{2}\varphi_{31}^{-},\eta_{31}\right),\end{aligned}$$

with A.I. Davydychev, hep-th/9908032

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right)=\frac{1}{2\varepsilon}\int\limits_0^{\varphi/2}\mathrm{d}\phi\left[1-\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon}\right]$$

ε-expansion:

$$\frac{1}{2}\sum_{j=0}^{\infty}\frac{(-\varepsilon)^j}{(j+1)!}\int\limits_0^{\varphi/2}\mathrm{d}\phi\;\ln^{j+1}\left(1+\frac{\tan^2\eta_{12}}{\cos^2\phi}\right)$$

In terms of Li_3, the ε-part was calculated in

U. Nierste, D. Müller and M. Böhm, Z. Phys. **C57** (1993) 605

The result for arbitrary ε can be presented in terms of Appell's hypergeometric function F_1,

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right)=\frac{1}{2\varepsilon}\left[\frac{\varphi}{2}-\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\cos^{2\varepsilon}\tau_0\; F_1\left(1,1,\varepsilon;\tfrac{3}{2}\middle|\;\sin^2\frac{\varphi}{2},\sin^2\frac{\tau}{2}\right)\right],$$

with $\cos\tau_0=\cos\eta\;\cos\frac{\tau}{2}$,

$$F_1(a,b,b',c|x,y)=\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a)_{j+l}\;(b)_j\;(b')_l}{(c)_{j+l}}\;\frac{x^j\;y^l}{j!\;l!}$$

Similar functions occurred in

O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Some special cases: L.G. Cabral-Rosetti, M.A. Sanchis-Lozano, hep-ph/0206081

An important formula (shift $\varepsilon \to 1+\varepsilon$, or $n \to n-2$):

$$F_1\left(1,1,\varepsilon;\tfrac{3}{2}\middle|\, x,y\right) = \left(1-\frac{y}{x}\right) F_1\left(1,1,1+\varepsilon;\tfrac{3}{2}\middle|\, x,y\right) + \frac{y}{x}\, {}_2F_1\left(\begin{array}{c} 1, 1+\varepsilon \\ 3/2 \end{array} \middle| \, y \right) .$$

It can be supplemented by a Kummer relation:

$$(1-2\varepsilon)\, {}_2F_1\left(\begin{array}{c} 1, \varepsilon \\ 3/2 \end{array} \middle| \, y \right) = 1 - 2\varepsilon(1-y)\, {}_2F_1\left(\begin{array}{c} 1, 1+\varepsilon \\ 3/2 \end{array} \middle| \, y \right) .$$

Using these relations recursively, we can express three-point function in $n+2j$ dimensions in terms of the three- and two-point functions in n dimensions, going from $\varepsilon - j$ to ε

To be compared with O.V. Tarasov, Phys. Rev. D54 (1996) 6479.

Special values of n: $n=3$ $(\varepsilon=\frac{1}{2})$:

We can use : $F_1(a,b,b';b+b'|\,x,y) = (1-y)^{-a}\;{}_2F_1\left(\begin{array}{c} a,b \\ b+b' \end{array}\middle|\, \frac{x-y}{1-y}\right)$,

$$ {}_2F_1\left(\begin{array}{c} 1,1 \\ 3/2 \end{array}\middle|\, z\right) = \frac{\arcsin\sqrt{z}}{\sqrt{z(1-z)}} \;, $$

We get $\omega\left(\tfrac{1}{2}\varphi,\eta\right)\Big|_{n=3} = \frac{\varphi}{2} - \frac{\pi}{2} + \kappa$, with $\cos\kappa = \sin\frac{\varphi}{2}\cos\eta$.

Collect results for all six triangles:

$$ \Omega^{(3;3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi \;. $$

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Special values of n: $n=2$ $(\varepsilon=1)$:

$$F_1\left(1,1,1;\tfrac{3}{2}\middle|\, x,y\right) = \frac{1}{x-y}\left[\frac{\sqrt{x}\;\arcsin\sqrt{x}}{\sqrt{1-x}} - \frac{\sqrt{y}\;\arcsin\sqrt{y}}{\sqrt{1-y}}\right] .$$

In this way, we get

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right)\Big|_{n=2} = \frac{\tau}{4}\sin\eta \; .$$

Collecting results for all six triangles, we get

$$\Omega^{(3;2)} = \frac{1}{2}\left(\tau_{12}\sin\eta_{12} + \tau_{23}\sin\eta_{23} + \tau_{13}\sin\eta_{13}\right) \; .$$

The two-point integral in two dimensions is proportional to $\tau/\sin\tau$
$\Rightarrow$ three-point integral in two dimensions is a combination of three two-point integrals, with coefficients proportional to $\sin\tau_{jl}\sin\eta_{jl}$.

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Special values of n: $n = 5$ $(\varepsilon = -\frac{1}{2})$:

In this case, we obtain

$$\omega\left(\tfrac{1}{2}\varphi, \eta\right)\Big|_{n=5} = -\frac{\varphi}{2} + \frac{\pi}{2} - \kappa + \frac{1}{2}\tan\eta\ \ln\left(\frac{1+\sin\frac{\tau}{2}}{1-\sin\frac{\tau}{2}}\right).$$

Collecting results for all six triangles, we obtain

$$\Omega^{(3;5)} = -\left(\psi_{12} + \psi_{23} + \psi_{31} - \pi\right) + \tan\eta_{12}\ \ln\left(\frac{m_1 + m_2 + \sqrt{k_{12}^2}}{m_1 + m_2 - \sqrt{k_{12}^2}}\right)$$
$$+ \tan\eta_{23}\ \ln\left(\frac{m_2 + m_3 + \sqrt{k_{23}^2}}{m_2 + m_3 - \sqrt{k_{23}^2}}\right) + \tan\eta_{13}\ \ln\left(\frac{m_1 + m_3 + \sqrt{k_{13}^2}}{m_1 + m_3 - \sqrt{k_{13}^2}}\right).$$

In other words, the five-dimensional three-point integral can be expressed in terms of the three-dimensional three- and two-point integrals.

Special values of n: $n = 4$ $(\varepsilon \to 0)$:

$$\int\limits_0^{\varphi/2} \mathrm{d}\phi \; \ln\left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right) = \tfrac{1}{2}\tau \; \ln\left(\frac{1+\sin\eta}{1-\sin\eta}\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi+\tau\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi-\tau\right) - \mathrm{Cl}_2\left(\varphi\right)$$

Compare with: P. Wagner, Indag. Math. 7 (1996) 527

After analytical continuation, corresponds to

G. 'tHooft and M. Veltman, Nucl. Phys. B153 (1979) 365

Analytic Continuation: Arbitrary Dimension

Consider $$\int\limits_0^{\varphi_0} \mathrm{d}\phi \, \left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} .$$

Substitute $z \Rightarrow e^{2\mathrm{i}\phi}$, so that $\cos^2\phi \Rightarrow \dfrac{(1+z)^2}{4z}$,

$$1 + \frac{\tan^2\eta}{\cos^2\phi} \Rightarrow \frac{(z+\rho)(z+1/\rho)}{(z+1)^2}, \quad \text{with} \quad \rho \equiv \frac{1-\sin\eta}{1+\sin\eta}$$

In this way, $$\int\limits_0^{\varphi_0} \mathrm{d}\phi \, \left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} \Rightarrow \frac{\mathrm{i}}{2} \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \, \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right]^{-\varepsilon},$$

with $z_0 \leftrightarrow e^{2\mathrm{i}\varphi_0}$.

Analytic Continuation: Expansion in ε

Expanding in ε, we get

$$Q_j \equiv \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \, \ln^j \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] .$$

The first term, $\mathcal{O}(1)$:

$$\begin{aligned} Q_1 &\equiv \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \, \ln \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] \\ &= \mathrm{Li}_2\left(-z_0\rho\right) + \mathrm{Li}_2\left(-z_0/\rho\right) - 2\mathrm{Li}_2\left(-z_0\right) + \tfrac{1}{2}\ln^2\rho \end{aligned}$$

Analytic Continuation: Expansion in ε (continued)

$$
\begin{aligned}
Q_2 \;&\equiv\; \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z}\; \ln^2\left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right] \\
&=\; \ln\rho \Bigg[2\mathrm{Li}_2\left(\frac{1-\rho}{1+z_0\rho}\right) + 2\mathrm{Li}_2\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) - 2\mathrm{Li}_2\left(\frac{\rho-1}{z_0+\rho}\right) - 2\mathrm{Li}_2\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \\
&\quad -\mathrm{Li}_2\left(\frac{1-\rho^2}{1+z_0\rho}\right) - \mathrm{Li}_2\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) + \mathrm{Li}_2\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{Li}_2\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right) \Bigg] \\
&\quad +4\mathrm{S}_{1,2}\left(\frac{1-\rho}{1+z_0\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) + 4\mathrm{S}_{1,2}\left(\frac{\rho-1}{z_0+\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \\
&\quad -\mathrm{S}_{1,2}\left(\frac{1-\rho^2}{1+z_0\rho}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) - \mathrm{S}_{1,2}\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right)
\end{aligned}
$$

Compare with: J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

ε-expansion: higher terms

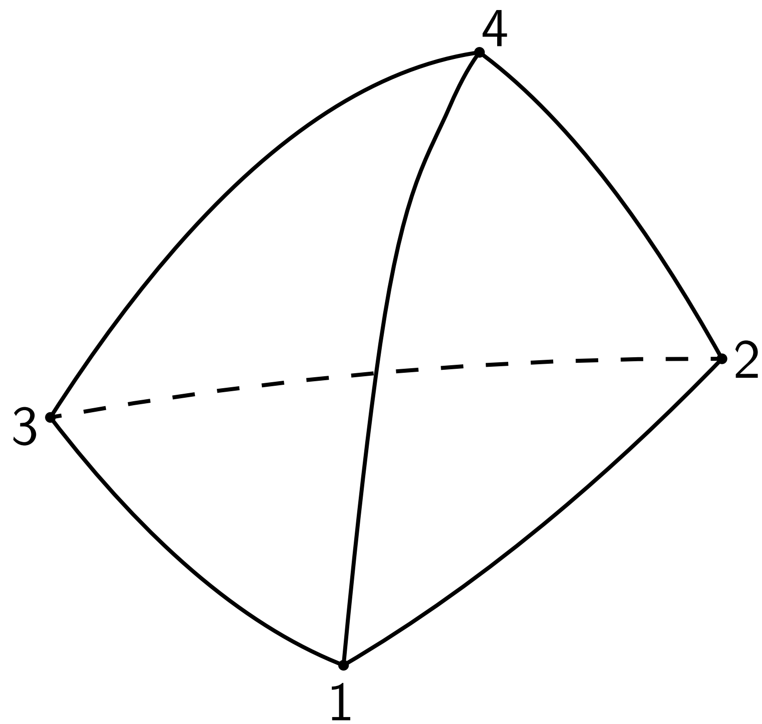
ε^2-term

$$Q_3 = \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln^3 \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right],$$

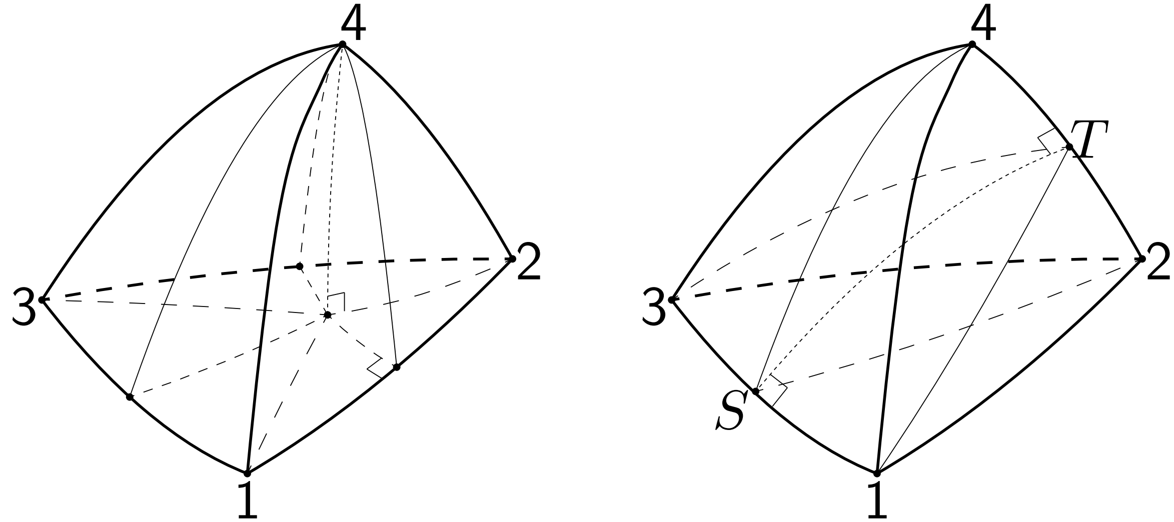
etc.

Some special cases considered in J.G. Körner, Z. Merebashvili, M. Rogal, Phys. Rev. D71 (2005) 054028

Geometrical approach: 4-point function



The spherical tetrahedon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