Applications and explicit solution of $d-$ shift recurrence relations

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Content

- reduction of tensor integrals
- $d-$ shift recurrence relations
- some applications of recurrence relations
- explicit solution for 1-loop case

(report on work in collaboration with J. Fleischer and F. Jegerlehner)

To evaluate L loop amplitudes one needs to calculate integrals

$$G^{(d)}(\{q_i q_k\}, \{m_s^2\}) =$$

$$\prod_{i=1}^{L} \int d^d k_i \prod_{j=1}^{N} P^{\nu_j}_{\overline{k}_j, m_j} \prod_{r=1}^{n_1} k_{1\mu_r} \ldots \prod_{s=1}^{n_L} k_{L\lambda_s},$$

where

$$P^j_{k,m} = \frac{1}{(k^2 - m^2 + i\epsilon)^j},$$

$$\overline{k}_j^\mu = \sum_{n=1}^{L} \omega_{jn} k_n^\mu + \sum_{m=1}^{E} \eta_{jm} q_m^\mu,$$

q_m - external momenta,

N - number of lines,

E - the number of external legs,

ω and red η are matrices of incidences

For one-loop graphs - Passarino - Veltman reduction solves the problem.

New method

A systematic method for evaluating tensor integral with any number of loops and legs proposed by O.V.T. (Phys.Rev. D54(1996) 6479):

$$\prod_{i=1}^{L} \int d^d k_i \prod_{j=1}^{N} P^{\nu_j}_{\overline{k}_j, m_j} \prod_{r=1}^{n_1} k_{1\mu_r} \ldots \prod_{s=1}^{n_L} k_{L\mu_s} =$$

$$T_{\mu_1 \ldots \lambda_L}(\{q_i\}, \{\partial_j\}, \mathbf{d}^+)$$

$$\times \prod_{i=1}^{L} \int d^d k_i \prod_{j=1}^{N} P^{\nu_j}_{\overline{k}_j, m_j} \Bigg|_{\{m_j^2 = m_{oj}^2\}},$$

where $T_{\mu_1 \ldots \lambda_L}$ - a polynomial type tensor operator,

$$\partial_j \equiv \frac{\partial}{\partial m_j^2}, \qquad \mathbf{d}^+ G^{(d)} = G^{(d+2)}.$$

First assume all lines have different masses, differentiation set masses to their original values.

To derive explicit formula for the operator T

— introduce L independent auxiliary vectors a_i

$$k_{1\mu_1}\ldots\ldots k_{L\lambda_s} = \frac{1}{i^{n_1+\ldots+n_L}} \frac{\partial}{\partial a_{1\mu_1}} \cdots \frac{\partial}{\partial a_{L\lambda_s}}$$

$$\times \exp\left[i(a_1k_1 + \ldots + a_Lk_L)\right]\Big|_{a_i=0} \cdot$$

— use the α- parametric representation.

$$T(\{q\},\{\partial\},\mathbf{d}^+) = \frac{e^{-iQ(\{\{q\},\alpha,\{0\})\rho}}{i^{n_1+\ldots+n_N}}$$

$$\times \prod_{r=1}^{n_1} \frac{\partial}{\partial a_{1\mu_r}} \cdots \prod_{s=1}^{n_L} \frac{\partial}{\partial a_{L\lambda_s}} e^{i[Q(\{q_j\},\alpha,\{a_j\})]\rho}\Bigg|_{\substack{a_j=0 \\ \alpha_j=i\partial_j \\ \rho=(\frac{i}{\pi})^L\mathbf{d}^+}} .$$

Q - polynom in $\alpha,$ a and q_iq_j.

Applying T gives a sum made of external momenta, $g_{\mu\nu}$ multiplied by scalar integrals with shifted **d**.

Advantages of the proposed tensor reduction:

- No contractions with external momenta and metric tensor and no solution of a linear system of equations are needed
- easy to select a particular tensor structure
- representation in terms of integrals with shifted d is compact
- T can be easily constructed by computer. For 2-3 loop tensor integrals of rank 3-4 in several seconds is needed using FORM.

But

Scalar integrals with different indices ν_j and shifts in d are to be evaluated! Method of the evaluation

Generalized recurrence relations

O.V. Tarasov, Phys.Rev., D54 (1996) 6479

Generalized recurrence relations

We start with the relation:

$$\prod_{i=1}^{L}\int d^dk_i \frac{\partial}{\partial k_{r\mu}}\left\{R_{\{\mu\}}\left(\{k\},\{q\}\right)\prod_{j=1}^{N}P^{\nu_j}_{\overline{k}_j,m_j}\right\}\equiv 0,$$

R - an arbitrary tensor polynomial.

Integration by parts for finding relations between Feynman integrals was used by Petersson (J.Math.Phys.6 (1965) 1955.) and by t'Hooft - Veltman (Nucl.Phys. B44 (1972)189). A systematic method was proposed by Tkachov and Chetyrkin (Nucl.Phys.B192 (1981) 159).

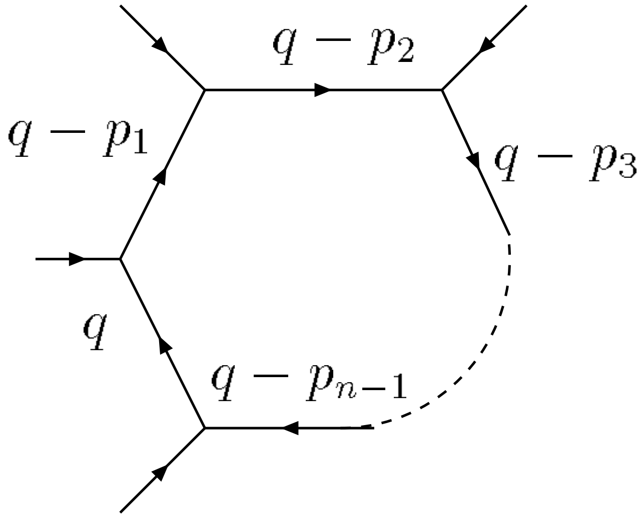
After differentiation two representation for scalar products can be used:

$$\text{a)}\quad k_iq_j=\frac{1}{2}(k_i^2+q_j^2-(k_i-q_j)^2),\qquad (Tkachov,\ Chetyrkin)$$

$$\text{b)}\quad q_{j\mu}\int k_{i\mu}d^dk_i\prod_{j=1}^{N}P^{\nu_j}_{\overline{k}_j,m_j}=$$

$$q_{j\mu}T_{\mu}(\{q\},\{\partial\},\mathbf{d}^+)\int d^dk_i\prod_{j=1}^{N}P^{\nu_j}_{\overline{k}_j,m_j}.\quad (Tarasov)$$

Combination of these relations gives many recurrence relations connecting integrals with different indices and shifts of d.



One-loop tensor integrals (see J. Fleischer, F. Jegerlehner and O.V.T Nucl.Phys. B566 (2000) 423)

$$I_{n,r}^{(d)} = \int \frac{d^d q}{\pi^{d/2}} \prod_{j=1}^{n} \frac{q_{\mu_1} \dots q_{\mu_r}}{c_j^{\nu_j}},$$

with

$$c_j = (q - p_j)^2 - m_j^2 + i\epsilon \quad \text{for} \quad j < n$$
$$c_n = q^2 - m_n^2 + i\epsilon.$$

can be written as a combination of scalar integrals

$$I_{n,r}^{(d)} = T_{\mu_1 \dots \mu_r}(\{p_s\}, \{\partial_j\}, \mathbf{d}^+) \int \frac{d^d q}{\pi^{d/2}} \prod_{j=1}^{n} \frac{1}{c_j^{\nu_j}},$$

where

$$T_{\mu_1 \dots \mu_r}(\{p_s\}, \{\partial_j\}, \mathbf{d}^+) = \frac{1}{i^r}$$
$$\times \prod_{j=1}^{r} \frac{\partial}{\partial a_{\mu_j}} \exp\left[i \left(\sum_{k=1}^{n-1} (a p_k) \alpha_k - \frac{1}{4} a^2 \right) \rho \right] \Bigg|_{\substack{a_j = 0 \\ \alpha_j = i\partial_j \\ \rho = i\mathbf{d}^+}} .$$

Three recurrence relations are needed to reduce scalar integrals to basic ones:

$$2\Delta_n \nu_j \mathbf{j}^+ I_n^{(d)} =$$
$$\sum_{k=1}^{n} (1+\delta_{jk}) \left(\frac{\partial \Delta_n}{\partial Y_{jk}}\right) \left[d - \sum_{i=1}^{n} \nu_i (\mathbf{k}^- \mathbf{i}^+ + 1)\right] I_n^{(d)},$$

$$G_{n-1} \nu_j \mathbf{j}^+ I_n^{(d+2)} = \left[(\partial_j \Delta_n) + \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n) \mathbf{k}^-\right] I_n^{(d)},$$

$$(d - \sum_{i=1}^{n} \nu_i + 1) I_n^{(d+2)} = \left[\frac{2\Delta_n}{G_{n-1}} + \sum_{k=1}^{n} \frac{(\partial_k \Delta_n)}{G_{n-1}} \mathbf{k}^-\right] I_n^{(d)}.$$

where $\mathbf{j}^{\pm}$ etc. shift the indices $\nu_j \to \nu_j \pm 1$

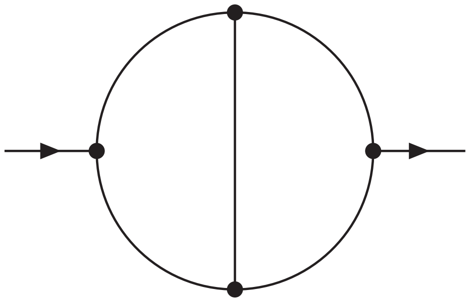
$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix},$$

$$Y_{ij} = -(p_i - p_j)^2 + m_i^2 + m_j^2,$$

$$G_{n-1} = -2^n \begin{vmatrix} p_1 p_1 & p_1 p_2 & \ldots & p_1 p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 p_{n-1} & p_2 p_{n-1} & \ldots & p_{n-1} p_{n-1} \end{vmatrix},$$

p_i, p_j -external momenta, m_j - the mass of the j-th line, ($p_n = 0$). Similar relations exist for integrals with zero Gramm determinants.

Two-loop propagator



Tensor integrals can be evaluated using relation:

$$\int\int \frac{d^dk_1 d^dk_2}{c_1^{\nu_1} c_2^{\nu_2} c_3^{\nu_3} c_4^{\nu_4} c_5^{\nu_5}} k_{1\mu_1} \ldots k_{1\mu_r} k_{2\lambda_1} \ldots k_{2\lambda_s}$$

$$= T_{\mu_1 \ldots \lambda_s}(q, \{\partial_j\}, \mathbf{d}^+) \int\int \frac{d^dk_1 d^dk_2}{c_1^{\nu_1} c_2^{\nu_2} c_3^{\nu_3} c_4^{\nu_4} c_5^{\nu_5}},$$

$$c_1 = k_1^2 - m_1^2 + i\epsilon, \qquad c_2 = k_2^2 - m_2^2 + i\epsilon$$
$$c_3 = (k_1 - q)^2 - m_3^2 + i\epsilon, \quad c_4 = (k_2 - q)^2 - m_4^2 + i\epsilon$$
$$c_5 = (k_1 - k_2)^2 - m_5^2 + i\epsilon$$

The recursive procedure (Tarasov, Nucl.Phys. B502 (1997) 455) transforms the diagram into a sum over 30 basic integrals:

$$I^{(d)}(q^2) = \sum_{j=1}^{30} R_j(q^2, \{m_i^2\}, d) I_j^{(d)}(q^2)$$

This algorithm implemented in a FORM packages (Tarasov and Veretin) and as a Mathematica package (Scharf and Mertig).

Some very interesting applications

—Evaluation of the 2-loop correction to the static potential in QCD (Y. Schröder, Phys.Lett. B447 (1999) 321).

—Reduction of tensor 2-loop massless box diagrams:

a) planar topology (V. Smirnov and O. Veretin, Nucl.Phys. B566 (2000) 469:

b) nonplanar topology (C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, hep-ph/0003261

These algorithms may be used to evaluate 2-loop correction to the Bhabha scaterring.

Explicit solution of $d-$ recurrences

We start with the relation connecting n point integrals with different space-time dimension

$$(d-n+1)I_n^{(\mathbf{d+2})} - \frac{2\Delta_n}{G_{n-1}} I_n^{(\mathbf{d})} = \sum_{k=1}^{n} \frac{\partial_k \Delta_n}{G_{n-1}} \mathbf{k}^- I_n^{(\mathbf{d})}.$$

If $n-1$ point functions are known then for n -point function we have nonhomogeneous **first order difference equation** w.r.t. $\mathbf{d}$. Methods of solution are well known. Redefintion

$$I_n^{(d)} = \frac{1}{\Gamma\left(\frac{d-n+1}{2}\right)} \left(\frac{\Delta_n}{G_{n-1}}\right)^{\frac{d}{2}} \overline{I}_n^{(d)}$$

gives simpler equation

$$\overline{I}_n^{(d+2)} = \overline{I}_n^{(d)}$$

$$+ \frac{\Gamma\left(\frac{d-n+1}{2}\right)}{2\Delta_n} \left(\frac{G_{n-1}}{\Delta_n}\right)^{\frac{d}{2}} \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- I_n^{(d)}.$$

Remarkable feature of the equation:

the number of momenta and masses does not complicate the equation: all is hidden in G_{n-1} and Δ_n.

Parametrizing $\mathbf{d}$ as

$$\mathbf{d} = \mathbf{2l} - \mathbf{2}\varepsilon.$$

where $\mathbf{l}$ is integer, solution for $\overline{I}_n^{(d)}$ can be written as

$$\overline{I}_n^{(2l-2\varepsilon)} = \sum_{r=0}^{l} \frac{\Gamma\left(r-1-\varepsilon-\frac{n-1}{2}\right)}{2\Delta_n} \left(\frac{G_{n-1}}{\Delta_n}\right)^{r-1-\varepsilon} \times \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- I_n^{(2r-2-2\varepsilon)} + \tilde{b}_n(\varepsilon),$$

where $\tilde{b}_n(\varepsilon)$ is $\mathbf{l}$ independent constant. By changing

$$\tilde{b}_n(\varepsilon) \to \tilde{b}_n(\varepsilon) - \sum_{r=0}^{\infty} \ldots,$$

taking the difference of 2 sums and changing $r \to r + l + 1$ the solution for $I^{(\mathbf{d})}$ can be rewritten as

$$I_n^{(d)} = b_n + \sum_{k=1}^{n} \left(\frac{\partial_k \Delta_n}{2\Delta_n}\right) \times \sum_{r=0}^{\infty} \left(\frac{d-n+1}{2}\right)_r \left(\frac{G_{n-1}}{\Delta_n}\right)^r \mathbf{k}^- I_n^{(d+2r)}.$$

The constant b_n can be determined from the asymptotic of $I_n^{(d)}$ at

$$d \to \infty$$

Asymptotic at $d \to \infty$

The asymptotic behaviour of $I_n^{(d)}$ at $d \to \infty$ can be found from it's parametric representation. Representing integrand as

$$\frac{1}{c_1 c_2 \ldots c_n} =$$

$$\Gamma(n) \int_0^1 \ldots \int_0^1 dx_1 \ldots dx_{n-1} \frac{x_1^{n-2} x_2^{n-3} \ldots x_{n-2}}{D^n},$$

with

$$D = [c_1 x_1 \ldots x_{n-1}$$
$$+ c_2 x_1 \ldots x_{n-2}(1 - x_{n-1}) + \ldots + c_n(1 - x_1)],$$

and integrating over q gives multiple integral

$$I_n^{(d)} = \Gamma\left(n - \frac{d}{2}\right) \int_0^1 \ldots \int_0^1 dx_1 \ldots dx_{n-1}$$
$$\times f(x_1, \ldots, x_{n-2})(h_n(\{p_j, m_s\}, x_1, \ldots, x_{n-1}))^{\frac{d}{2} - n},$$

with h_n being polynomial in x_j, momenta and masses.

This parametric integral is multiple Laplace type integral which can be written as

$$F(\lambda) = \int_\Omega f(x) \exp[\lambda S(x)] dx,$$

where $\lambda = d - n/2$, and $S(x) = \ln h_n$ is real function.

If the absolute maximum of $S(x)$ in Ω is achieved only at the interior point $x = \overline{x}$, then at $\lambda \to \infty$

$$F(\lambda) \sim \exp[\lambda S(\overline{x})] \sum_{r=0}^{\infty} a_k \lambda^{-r-\frac{k}{2}}.$$

The leading term of the expansion is

$$F(\lambda) = \exp[\lambda S(\overline{x})]\, (2\pi/\lambda)^{\frac{k}{2}}\, \frac{f(\overline{x}) + O(\lambda^{-1})}{\sqrt{|\det S''_{xx}(\overline{x})|}}.$$

If function h_n reach its maximum value on the boundary of the integration region then

$$F(\lambda) \sim \exp[\lambda S(\overline{x})] \sum_{r=0}^{\infty} a_k \lambda^{-r-\frac{k+1}{2}}.$$

The position of the maximal value of the integrand depends on momenta and masses.

Main contribution to the integral will come from the maximum of h_n. It can be either on the boundary of the integration region or inside interior region. To find the position $\overline{x}$ of the maximum, solve equation:

$$\frac{\partial h_n}{\partial x_i} = 0.$$

For the case when maximum is in the interior region then

$$\overline{x}_i = \frac{\sum_{k=1}^{n-i} \partial_k \Delta_n}{\sum_{j=1}^{n-i+1} \partial_j \Delta_n},$$

and the integrand is

$$f(\overline{x}_1, \ldots, \overline{x}_{n-2}) \left(h_n(\{p_j, m_s\}, \overline{x}_1, \ldots, \overline{x}_{n-1})\right)^{\frac{d}{2}-n}$$
$$= \left(-\frac{\Delta_n}{G_{n-1}}\right)^{\frac{d}{2}-n} \frac{1}{G_{n-1}^{\frac{n}{2}}}.$$

When integrand of $I_n^{(d)}$ has maximum inside the integration region, the result for $I_n^{(d)}$ reads

$$I_n^{(\mathbf{d})} = (-1)^n (2\pi)^{\frac{n}{2}} \frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-n+1}{2}\right)} \frac{\Delta_n^{\frac{d-n-1}{2}}}{(-G_{n-1})^{\frac{d-n}{2}}}$$
$$+ \sum_{k=1}^{n} \left(\frac{\partial_k \Delta_n}{2\Delta_n}\right) \sum_{r=0}^{\infty} \left(\frac{d-n+1}{2}\right)_r \left(\frac{G_{n-1}}{\Delta_n}\right)^r \mathbf{k}^- I_n^{(\mathbf{d+2r})},$$

where $(a)_r \equiv \Gamma(r+a)/\Gamma(a)$ is the Pochhammer symbol.

If for some kinematical configuration $I_n^{(d)}$ has an imaginary part then

$$h_n < 0.$$

One can represent $I_n^{(d)}$ as

$$I_n^{(d)} = \Gamma\left(n - \frac{d}{2}\right)$$
$$\times \left\{ \int \{dx\} \theta(h_n) f(\{x\}) (h_n(\{p_j, m_s\}, \{x\}))^{\frac{d}{2}-n} \right.$$
$$+ \cos\frac{\pi}{2}(d-2n)$$
$$\times \sum_j \int_{\Omega_j^-} \{dx\} f(\{x\}) \left| h_n(\{p_j, m_s\}, \{x\}) \right|^{\frac{d}{2}-n}$$
$$+ i \sin\frac{\pi}{2}(d-2n)$$
$$\left. \times \sum_j \int_{\Omega_j^-} \{dx\} f(\{x\}) \left| h_n(\{p_j, m_s\}, \{x\}) \right|^{\frac{d}{2}-n} \right\}.$$

where Ω_j^- are subdomains of integration region where $h_n < 0$, and find asymptotic at $\mathbf{d} \to \infty$ for real and imaginary parts. Real and imaginary parts are again integrals of the Laplace type.

Two-point function

For our convenience we will label lines of $I_2^{(d)}$ as i, j. By using expression for the tadpole integral $I_1^{(d)}$

$$I_1^{(d)} = -\frac{\Gamma\left(1-\frac{d}{2}\right)}{(m_i^2)^{1-\frac{d}{2}}}.$$

from the general result we obtain

$$\frac{2\lambda_{ij} I_2^{(d)}}{\Gamma\left(1-\frac{d}{2}\right)} = b_2 \quad + \quad \frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(-\frac{m_j^2 G_1}{\lambda_{ij}}\right)^r$$

$$+ \quad \frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(-\frac{m_i^2 G_1}{\lambda_{ij}}\right)^r.$$

where

$$G_1 = -4p_{ij}^2 = -4(p_i - p_j)^2.$$

The parametric formula for $I_2^{(2)}$ is

$$I_2^{(d)} = \Gamma\left(2-\frac{d}{2}\right) \int_0^1 dx_1 \, h_2^{\frac{d}{2}-2}.$$

The integrand

$$h_2 = p_{12}^2 x_1^2 - x_1(p_{12}^2 - m_1^2 + m_2^2) + m_2^2$$

has extremum at

$$\overline{x}_1 = \frac{p_{12}^2 - m_1^2 + m_2^2}{2p_{12}^2},$$

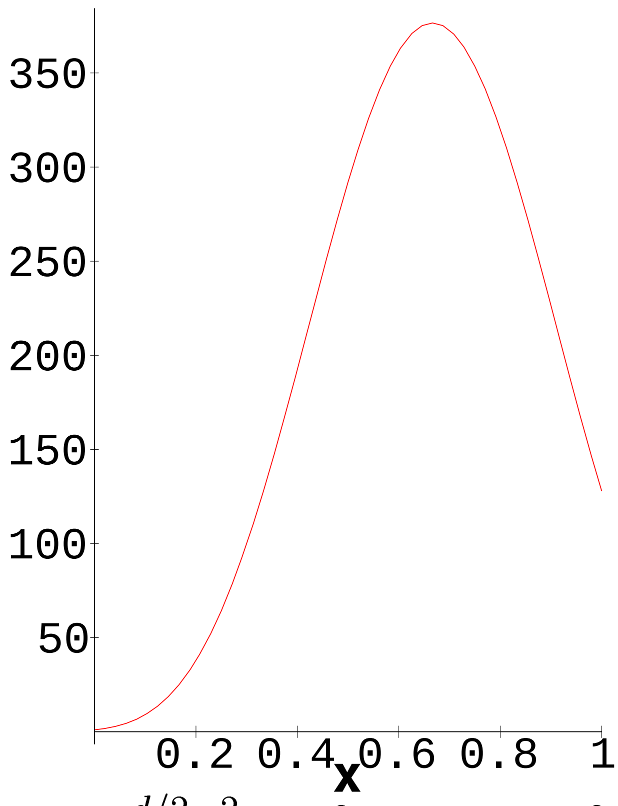


Fig. 2: Function $h_2^{d/2-2}$ at $p_{12}^2 = -3, m_1^2 = 2, m_2^2 = 1,$ $d = 18$.

From the analysis of I_2^d at large d the term b_2 can be found for arbitrary momenta and masses. Taking into account that infinite sums are just hypergeometric functions ${}_2F_1$ we obtain

$$\frac{2\lambda_{ij}\; I_2^{(d)}}{\Gamma\left(1-\frac{d}{2}\right)} =$$

$$\frac{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} r_{ij}^{\frac{d-2}{2}} \left[\frac{\partial_i \lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}} + \frac{\partial_j \lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}\right]$$

$$+\frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}}\, {}_2F_1\left[\begin{matrix} 1, \frac{d-1}{2}\,; & \frac{m_j^2}{r_{ij}} \\ \frac{d}{2}\,; & \end{matrix}\right]$$

$$+\frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}}\, {}_2F_1\left[\begin{matrix} 1, \frac{d-1}{2}\,; & \frac{m_i^2}{r_{ij}} \\ \frac{d}{2}\,; & \end{matrix}\right].$$

where

$$r_{ij} = -\frac{\lambda_{ij}}{g_{ij}} = -\frac{\Delta_2}{G_1}.$$

Three -point function

For convenience we will label lines of integral $I_3^{(d)}$ as i, j, k, and introduce notation

$$r_{ijk} = -\frac{\lambda_{ijk}}{g_{ijk}} = -\frac{\Delta_3}{G_2}.$$

Formula for $I_3^{(d)}$ includes summation over ${}_2F_1$ coming from $I_2^{(d)}$. Sum over $I_2^{(d)}$ gives the Appell hypergeometric functions $F_3((d-2)/2, 1, 1, 1/2, d/2; x, y)$

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \\ \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n,$$

which may be reduced to F_1 by means of relation

$$F_3(\alpha, \alpha', \beta, \beta', \alpha+\alpha'; x, y) = \\ (1-y)^{-\beta'} F_1\left(\alpha, \beta, \beta', \alpha+\alpha'; x, \frac{y}{y-1}\right).$$

The result for $I_3^{(d)}$ then reads

$$\frac{\lambda_{ijk}}{\Gamma\left(2-\frac{d}{2}\right)} I_3^{(d)} = 2^{\frac{3}{2}}\pi\,\sqrt{g_{ijk}}\, r_{ijk}^{\frac{d-2}{2}}$$
$$+\theta_{ijk}\,\partial_k\lambda_{ijk} + \theta_{kij}\,\partial_j\lambda_{ijk} + \theta_{jki}\,\partial_i\lambda_{ijk},$$

where

$$\lambda_{ij}\,\theta_{ijk} = -\left[\frac{\partial_i\lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}} + \frac{\partial_j\lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}\right] r_{ij}^{\frac{d-2}{2}}$$
$$\times\frac{\sqrt{\pi}\,\Gamma\left(\frac{d-2}{2}\right)}{4\Gamma\left(\frac{d-1}{2}\right)}\,{}_2F_1\left[\begin{matrix}1,\frac{d-2}{2};\\ \frac{d-1}{2};\end{matrix}\;\frac{r_{ij}}{r_{ijk}}\right]$$
$$+\frac{(m_i^2)^{\frac{d-2}{2}}}{2(d-2)}\frac{\partial_j\lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}} F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};\frac{m_i^2}{r_{ijk}},\frac{m_i^2}{r_{ij}}\right)$$
$$+\frac{(m_j^2)^{\frac{d-2}{2}}}{2(d-2)}\frac{\partial_i\lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}} F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};\frac{m_j^2}{r_{ijk}},\frac{m_j^2}{r_{ij}}\right).$$

Appell hypergeometric function F_1 defined as

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}\,m!\,n!}x^m y^n,$$

and in our case it admits simple integral representation:

$$F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};x,y\right) = \frac{d-2}{2}\int_0^1\frac{u^{\frac{d-4}{2}}}{(1-xu)\sqrt{1-yu}}du.$$

For the 3-point function

$$h_3 = -x_1 x_2 (1-x_1) p_{13}^2 - x_1^2 x_2 (1-x_2) p_{12}^2$$
$$-x_1(1-x_1)(1-x_2) p_{23}^2$$
$$+ x_1 x_2 m_1^2 + x_1(1-x_2) m_2^2 + (1-x_1) m_3^2.$$

h_3 has maximum at

$$\overline{x}_1 = \frac{\partial_1 \Delta_3 + \partial_2 \Delta_3}{\partial_1 \Delta_3 + \partial_2 \Delta_3 + \partial_3 \Delta_3},$$
$$\overline{x}_2 = \frac{\partial_1 \Delta_3}{\partial_1 \Delta_3 + \partial_2 \Delta_3}$$

if

$$\frac{\partial^2 h_3}{\partial x_2 \partial x_2} = 2 x_1^2 p_{12}^2 < 0,$$

$$\frac{\partial^2 h_3}{\partial x_1 \partial x_1} \frac{\partial^2 h_3}{\partial x_2 \partial x_2} - \left(\frac{\partial^2 h_3}{\partial x_1 \partial x_2} \right)^2 = \frac{(\partial_1 \Delta_3 + \partial_2 \Delta_3)^2}{(-2G_2)} > 0.$$

that is equivalent to

$$p_{12}^2 < 0,$$
$$G_2 < 0.$$

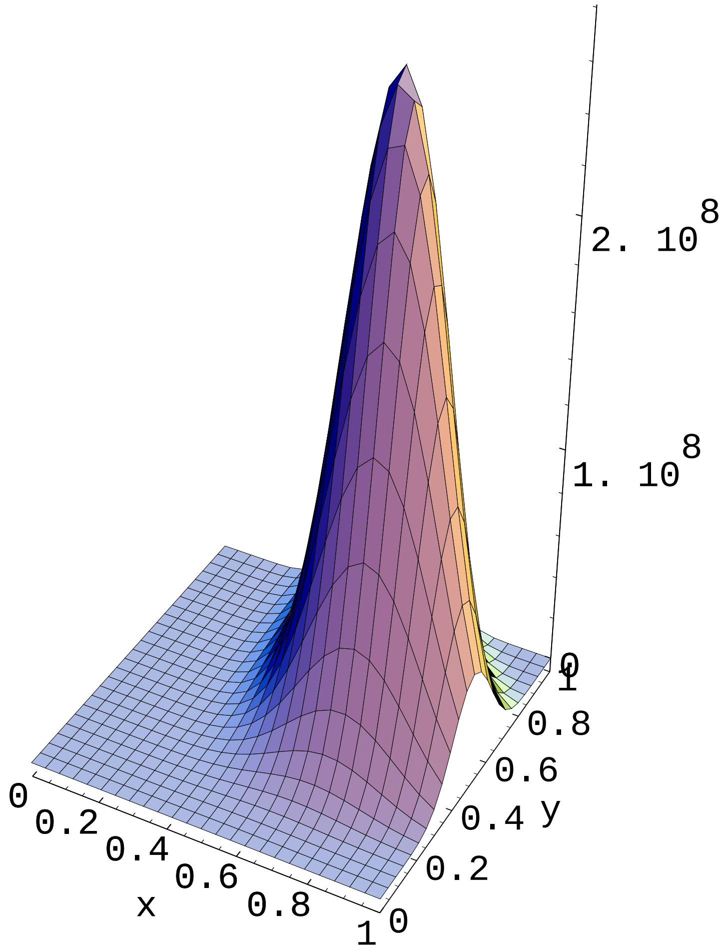


Fig.: Function $x_1 h_3^{d/2-3}$ at

$$p_{13}^2 = -10,\ p_{23}^2 = -5,\ p_{12}^2 = -9, m_i^2 = 1, d = 36.$$

The evaluation of b_3 in other kinematical regions is strightforward.

Four -point function

The result for the 4-point function reads

$$\frac{\lambda_{ijkl}}{\Gamma\left(2-\frac{d}{2}\right)} I_4^{(d)} = -4\pi^{\frac{3}{2}}\, \sqrt{g_{ijkl}}\; r_{ijkl}^{\frac{d-3}{2}} \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-3}{2}\right)}$$

$$+\phi_{ijkl}\,\partial_l \lambda_{ijkl} + \phi_{lijk}\,\partial_k \lambda_{ijkl} + \phi_{klij}\,\partial_j \lambda_{ijkl} + \phi_{jkli}\,\partial_i \lambda_{ijkl},$$

where

$$\phi_{ijkl} = -\frac{\pi\,\sqrt{2}}{\sqrt{g_{ijk}}}\, r_{ijkl}^{\frac{d-4}{2}}\;\; {}_2F_1\left[\begin{array}{cc} 1, \frac{d-3}{2}\,; & \frac{r_{ijk}}{r_{ijkl}} \\ \frac{d-2}{2}\,; & \end{array}\right]$$

$$+\, terms \quad with \;\; F_1 \;\; and \;\; F_S,$$

where

$$r_{ijkl} = -\frac{\lambda_{ijkl}}{g_{ijkl}} = -\frac{\Delta_4}{G_3}.$$

Function F_1 was already encountered in the three-point function and F_S is **Lauricella - Saran** function depending on three variables and admitting simple integral representation

$$F_S = \frac{\Gamma\left(\frac{d}{2}\right)\,(y-z)^{-\frac{1}{2}}}{\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{3}{2}\right)} \int_0^1 \frac{\arcsin\sqrt{\frac{(y-z)t}{1-tz}}\,(1-t)^{\mathbf{\frac{d-5}{2}}}}{(1-x+tx)\sqrt{1-ty}}\, dt.$$

Two-loop propagator integral

Relation between two-loop propagator master integral with different dimensions (Tarasov, Nucl.Phys. B502 (1997) 455) reads

$$\Delta F_{11111}^{(\mathbf{d-2})} = -q^2(d-3)(d-4)F_{11111}^{(\mathbf{d})}$$
$$+(P_1\mathbf{1}^- + P_2\mathbf{2}^- + P_3\mathbf{3}^- + P_4\mathbf{4}^- + P_5\mathbf{5}^-)F_{11111}^{(d)},$$

where

$$\Delta = q^4 m_5^2 - [(m_1^2+m_2^2+m_3^2+m_4^2-m_5^2)m_5^2$$
$$-(m_4^2-m_3^2)(m_2^2-m_1^2)]q^2 + (m_3^2-m_1^2)(m_4^2-m_2^2)m_5^2$$
$$+(m_1^2-m_2^2-m_3^2+m_4^2)(m_1^2m_4^2-m_2^2m_3^2),$$

and P_i are polynomials in $\{m_i^2\}$, q^2, d and operators $\mathbf{1}^+,\ldots,\mathbf{5}^+$. The polynomial P_i does not depend on the operator $\mathbf{i}^+$. The final formula connecting $F_{11111}^{(d)}$ and $F_{11111}^{(d-2)}$ with all masses arbitrary is rather long.

Again we have first order difference equation.
Solution is similar to the one loop case.

Integrals at the right hand side are known in terms of hypergeometric functions but the result is rather long. There are 8 sun-set functions (each gives 4 Lauricella functions), 4 integrals with four lines plus 17 other functions.

When some masses are equal or some are zero then the result significantly simplifies.

Conclusions

- $\mathbf{d}$- shift recurrence relations turns out to be rather useful in evaluating tensor integrals
- solution of $\mathbf{d}$ recurrences gives new, compact hypergeometric representation for Feynman integrals
- expansion of Feynman integrals (developed by product) at $\mathbf{d} \to \infty$ may be used for approximate evaluation of Feynman integrals. $\frac{\mathbf{1}}{\mathbf{d}}$ expansion works quite well in quantum mechanics and in lattice theories.