The relation between the $\overline{\mathrm{MS}}$ and the on-shell quark mass at order α_s^3

Matthias Steinhauser, University of Hamburg

(in collaboration with K.G. Chetyrkin,
University of Karlsruhe)

K.G. Chetyrkin and M. Steinhauser
– *Phys. Rev. Lett.* **83** (1999) 4001
– hep-ph/9911434, *Nucl. Phys.* **B** (in press)

INTRODUCTION

QUARK masses

- parameter in QCD Lagrangian $(m\bar{\Psi}\Psi)$
- not directly measurable
- pole masses, on-shell masses, running masses, ...
- in general: scheme and scale dependent

On-shell mass M

+ defined through pole of propagator (pole mass)

+ scheme and scale independent

+ useful for threshold calculations

– BUT: NOT short distance mass, receives sizeable contributions from large distances $\longrightarrow$ large coefficients in perturbation theory when M is used for parametrization [Bigi et al. (1994), Beneke, Braun (1994)]

$\overline{\text{MS}}$ mass m

+ short-distance mass

+ convenient in practical calculations

+ 4-loop anomalous dimension γ_m available
  [Chetyrkin (1997), Larin et al. (1997)]

– BUT:
  – NOT suitable for threshold phenomena
  – $m = m(\mu)$: physical meaning of μ ?

MOM scheme, RI scheme, . . .

$\rightarrow$ NLC, FMC: $\sigma(e^+e^- \rightarrow t\bar{t})$ close to threshold $\longrightarrow$ determination of top quark mass

$\rightarrow$ Υ sum rules $\longrightarrow$ determination of bottom quark mass

$\rightarrow$ after determination of mass $\longrightarrow$ conversion to $\overline{\mathrm{MS}}$ mass needed

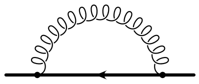
$\rightarrow$ Theoretical predictions for $\sigma(e^+e^- \rightarrow t\bar{t})$ and Υ sum rules available at NNLO $\longrightarrow$ $\overline{\mathrm{MS}}$–on-shell relation needed up to order α_s^3

TECHNIQUE

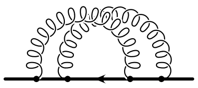
$$m(\mu) = z_m(\mu)\, M$$

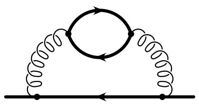
$\overline{\text{MS}}$ mass   on-shell mass

$$z_m(\mu) = 1 + \frac{\alpha_s(\mu)}{\pi} C_F z_m^F + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \times$$

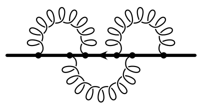


$$\left[C_F^2 z_m^{FF} + C_F C_A z_m^{FA} + C_F T n_l z_m^{FL} + C_F T z_m^{FH} \right]$$



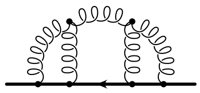


$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \Big[C_F^3 z_m{}^{FFF} + C_F^2 C_A z_m{}^{FFA}$$



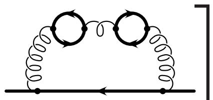
$$+ C_F C_A^2 z_m{}^{FAA} + C_F^2 T n_l z_m{}^{FFL} + C_F^2 T z_m{}^{FFH}$$

$$+ C_F C_A T n_l z_m{}^{FAL} + C_F C_A T z_m{}^{FAH}$$



$$+ C_F T^2 n_l^2 z_m{}^{FLL} + C_F T^2 n_l z_m{}^{FLH}$$

$$+ C_F T^2 z_m{}^{FHH} \Big]$$



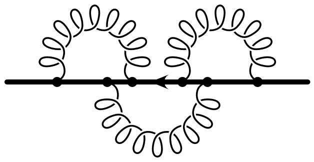
131 diagrams at 3-loop level

Requirement for on-shell mass, M:
inverse fermion propagator must have a pole

$$\left[S_F(q) \right]^{-1} \Bigg|_{q^2=M^2} = 0$$

$\longrightarrow$ 1-, 2- and 3-loop on-shell integrals needed
$\longrightarrow$ not available in the literature

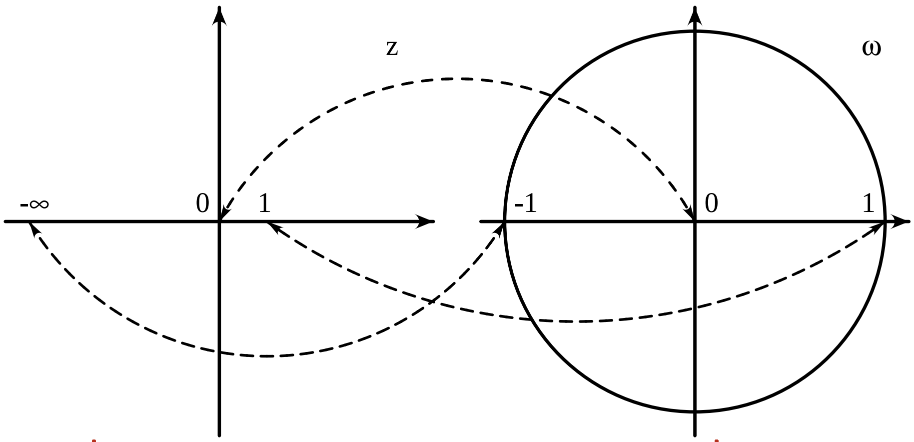
Our Method



- evaluate diagrams for

  $q^2 \ll M^2 \longrightarrow$ Taylor expansion
  $q^2 \gg M^2 \longrightarrow$ large momentum expansion
  7 terms in each region computed

- apply conformal mapping:



z plane $\qquad\qquad\qquad$ ω plane

$$z = \frac{q^2}{M^2} = 1 \qquad \longrightarrow \qquad \omega = 1$$

- Padé approximation (in ω or z): combine data from different kinematical regions

$$[n/m](\omega) = \frac{a_0 + a_1\omega + \ldots + a_n\omega^n}{b_0 + b_1\omega + \ldots + b_m\omega^m} = f(\omega) + \mathcal{O}(\omega^{n+m+1})$$

[Baikov, Broadhurst, Fleischer, Tarasov, Davydychev, ...]

RESULTS

$\overline{\text{MS}}$–on-shell mass relation at 1 and 2 loops

$$\frac{m(\mu = M)}{M} = 1 + \frac{\alpha_s(\mu)}{\pi} C_F z_m^F + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \times$$
$$\left[C_F^2 z_m^{FF} + C_F C_A z_m^{FA} + C_F T n_l z_m^{FL} + C_F T z_m^{FH} \right]$$
$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 z_m^{(3)} \qquad \longleftarrow \text{ later}$$

n	P.A.	F	FF	FA	FL
5	[4/5]	−1.0010	−0.4881	−3.3813	1.5687
5	[5/4]	−1.0008	−0.4870	−3.3936	1.5676
5	[5/5]	−1.0006	−0.4862	−3.3699	1.5663
5	[6/5]	—	−0.4884	−3.3972	—
	...				
6	[5/6]	−1.0002	−0.4866	−3.4084	1.5640
6	[5/7]	−1.0003	−0.4938	−3.3824	1.5646
6	[6/4]	−1.0007	−0.4894	−3.3830	1.5667
6	[6/6]	−1.0003	−0.4897	—	1.5646
result		−1.0005(8)	−0.49(1)	−3,4(1)	1.566(5)
exact		−1	−0.5105	−3.330	1.562

[Gray, Broadhurst, Grafe, Schilcher (1990)]

z^n terms used as input

Gauge dependence (F, FF, FA) $\left(i\frac{-g^{\mu\nu}+\xi\frac{q^\mu q^\nu}{q^2}}{q^2+i\epsilon}\right)$

$\xi = -5$

n	P.A.	F	FF	FA
5	[5/5]	−0.9990	−0.4673	−3.3145
5	[5/6]	−0.9996	−0.4617	−3.2547
5	[6/4]	−0.9989	−0.4629	−3.2604
	...			
6	[5/7]	−0.9994	−0.4721	−3.2672
6	[6/4]	−0.9989	−0.4670	−3.2513
6	[6/6]	−0.9994	−0.4518	−3.2837
exact		−1	−0.5105	−3.330

$\longrightarrow$ much worse

$\xi = -2$

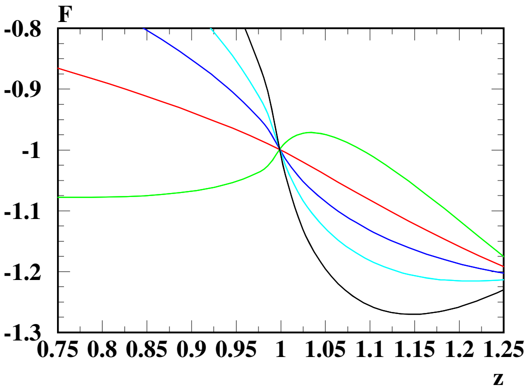
n	P.A.	F	FF	FA
5	[5/5]	−1.0000	−0.5209	−3.3499
5	[5/6]	−1.0000	−0.5100	−3.3521
5	[6/4]	−1.0000	−0.5055	−3.3517
	...			
6	[5/7]	−1.0000	−0.5114	−3.3472
6	[6/4]	−1.0000	−0.5083	−3.3497
6	[6/6]	−1.0000	−0.5112	−3.3557
exact		−1	−0.5105	−3.330

$\longrightarrow$ now the agreement is perfect

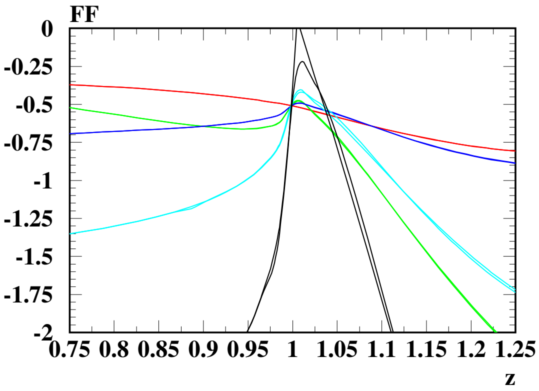
Criterion for $\xi = -2$ needed!

It is known [Landau, Abrikosov, Khalatnikov (1954)] that for $\xi = -2$ the QED fermion propagator gets infra-red finite (in leading-log approximation) $\longrightarrow$ best results are obtained for $\xi = -2$

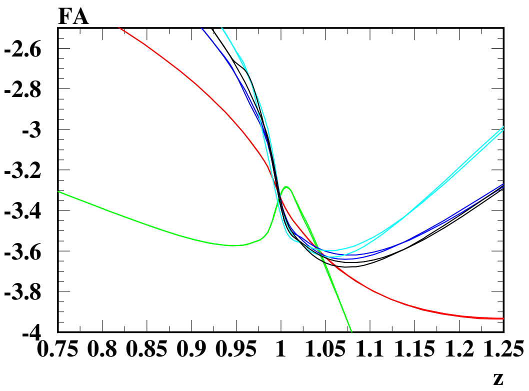
C_F



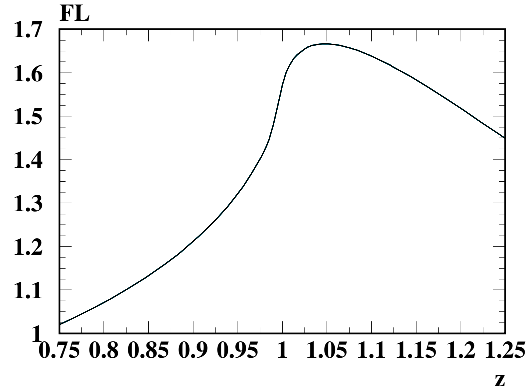
C_F^2



$C_A C_F$



$C_F T n_l$



red: $\xi = -2$ others: $\xi = 5, 2, 0, -5$

Of physical interest is only the fixed-n_l-result:

1. add results for moments (expansions for small and large q^2)

2. perform approximation procedure

$$C_F^2 z_m^{FF} + C_F C_A z_m^{FA} + C_F T n_l z_m^{FL}$$

n_l	0	1	2
$\xi = 0$	$-14.4(6)$	$-13.3(6)$	$-12.3(4)$
$\xi = -2$	$-14.3(2)$	$-13.2(2)$	$-12.2(2)$
exact	-14.2287	-13.1874	-12.146

n_l	3	4	5
$\xi = 0$	$-11.2(5)$	$-10.2(3)$	$-9.1(3)$
$\xi = -2$	$-11.1(1)$	$-10.07(3)$	$-9.02(1)$
exact	-11.1046	-10.0633	-9.02188

$\overline{\text{MS}}$–on-shell mass relation at 3 loops

- terms up to $z^6 = \left(\frac{q^2}{M^2}\right)^6$ and $\left(\frac{1}{z}\right)^6 = \left(\frac{M^2}{q^2}\right)^6$ available

- only QED part could be computed for general ξ

- other results are only available for $\xi = 0$
  gluon propagator: $i\frac{-g^{\mu\nu}+\xi\frac{q^\mu q^\nu}{q^2}}{q^2+i\epsilon}$

$\longrightarrow$ $\xi = 0$ is close to $\xi = -2$

$\longrightarrow$ physically interesting: results for fixed n_l:
1. add results for moments (expansions for small and large q^2)
2. perform approximation procedure
   $\longrightarrow$ Padé results are much more stable
3. perform fit to get n_l dependence

Final result

$$\frac{m(M)}{M} = 1 - 1.333\frac{\alpha_s(M)}{\pi} + \left(\frac{\alpha_s(M)}{\pi}\right)^2 [-14.33 + 1.041 n_l] + \left(\frac{\alpha_s(M)}{\pi}\right)^3 \left[-202(5) + 27.3(7) n_l - 0.653 n_l^2\right]$$

- error below 3% (from spread of Padé results)

Comparison of $M/m(m)$ with FAC, PMS, large-β_0

n_l	our results	FAC	PMS	"large-β_0"
2	143(3)	152.71	153.76	137.23
3	119(3)	124.10	124.89	118.95
4	96(2)	97.729	98.259	101.98
5	75(2)	73.616	73.903	86.318

FAC/PMS: [Chetyrkin, Kniehl, Sirlin (1997)]
large-β_0: [Beneke, Braun (1995)]

FAC: Fastest Apparent Convergence [Grunberg (1980)]
PMS: Principle of Minimal Sensitivity [Stevenson (1981)]

Check: Comparison with recent computation: [Melnikov, v. Ritbergen (1999)]

colour factor	our result	Mel/vRit
C_F^3	2.1(4)	2.05883
$C_F^2 C_A$	−2.0(4)	−2.34231
$C_F^2 T n_l$	0.8(2)	0.841686
$C_F^2 T n_h$	2.4(1)	2.49364
$C_F C_A^2$	−16.4(3)	−15.8503
$C_F C_A T n_l$	13(1)	13.0990
$C_F C_A T n_h$	−1.51(5)	−1.52186
$C_F T^2 n_l^2$	−1.98(6)	−1.95807
$C_F T^2 n_l n_h$	−0.0669(6)	−0.0667304
$C_F T^2 n_h^2$	−0.1922(0)	−0.192241

fixed-n_l results are in perfect agreement:

	$m(M)/M$	
n_l	our result	Mel/vRit
0	−202(5)	−198.7
1	−176(4)	−172.4
2	−150(3)	−147.5
3	−126(3)	−123.8
4	−103(2)	−101.5
5	−82(2)	−80.40

- $\sigma(e^+e^- \to t\bar{t})$: pole mass as parameter $\longrightarrow$ strong dependence of peak position; LO $\longrightarrow$ NLO $\longrightarrow$ NNLO; infra-red sensitivity

- idea: the static potential, V(r), has the same kind of uncertainties $\longrightarrow$ consider $2M + V(r)$
- M_{PS}, potential mass [Beneke (1998)]
  M_{1S} [Hoang, Teubner (1999)]

- use M_{PS}, resp., M_{1S} to parametrize $\sigma(e^+e^- \to t\bar{t})$ $\longrightarrow$ peak position stable $\longrightarrow$ M_{PS}, resp., M_{1S} are determined from experiment
  pole mass only needed in intermediate steps

- $\overline{\text{MS}}$ mass (needed for "non-threshold" quantities) computed via $\mathcal{O}(\alpha_s^3)$ relation

$$
\begin{tikzcd}
\boxed{\text{pole mass}} \arrow[rr, leftrightarrow, "\text{needed}"] \arrow[dr, leftrightarrow, "\text{known}"'] & & \boxed{\overline{\text{MS}}\text{ mass}} \arrow[dl, leftrightarrow, "\text{wanted}"] \\
& \boxed{\text{new short-distance mass}} &
\end{tikzcd}
$$

NNLO analysis needs $\overline{\text{MS}}$—on-shell relation at order α_s^3

Top quark

potential mass; $m_t(m_t) = 165.0$ GeV

$$\begin{array}{rcl} & & \qquad\qquad\quad \alpha_s^1 \qquad \alpha_s^2 \qquad \alpha_s^3 \\ M_{t,PS}(20\text{ GeV}) & = & [165.0 + 6.7 + 1.2 + 0.28]\text{ GeV} \\ M_t & = & [165.0 + 7.6 + 1.6 + 0.51]\text{ GeV} \end{array}$$

$\longrightarrow$ $M_{t,PS} - m_t(m_t)$ relation more accurate

$\longrightarrow$ "0.28" same magnitude as experimental error

[Beneke (1998)]

1S-mass; $M_{t,1S} = 175.0$ GeV

$$m_t(m_t) = [175.0 - 7.60 - 0.97 - 0.14]\text{ GeV}$$

$\longrightarrow$ large β_0 result: $0.14 \to 0.23$

$\longrightarrow$ error dominated by $\delta\alpha_s(M_Z)$

[Hoang, Teubner (1999)]

Bottom quark

1S-mass; $M_{b,1S} = 4.71$ GeV

$$m_b(m_b) = [4.71 - 0.40 - 0.11 - 0.03]\text{ GeV}$$

[Hoang (1999)]

CONCLUSIONS

- 3-loop, $\mathcal{O}(\alpha_s^3)$ term in the $\overline{\text{MS}}$–on-shell mass relation evaluated
- expand diagrams for $q^2 \ll M^2$ and $q^2 \gg M^2$; use conformal mapping and Padé approximation
- error (from approximation): 2% – 3%
- significant reduction of error in determination of $\overline{\text{MS}}$ mass
- $\overline{\text{MS}}$–on-shell relation computed with two methods $\longrightarrow$ very good agreement
- PMS, FAC work well $\longrightarrow$ $\mathcal{O}(\alpha_s^4)$ (!?)
- method can be used to get fermion propagator for arbitrary q^2/M^2 ($\longrightarrow$ lattice)