Structure functions: Reconstruction from Mellin moments

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1. **Introduction**
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S.M. and J.Vermaseren,
`hep-ph/9912355`, Nucl.Phys.B 573 (2000) 853

Introduction

- Deep inelastic lepton-hadron scattering:
  - unique information about proton $\longrightarrow$ constraints on PDF
  - measurement of α_s at scale Q

- NNLO analysis for DIS reactions of basic importance
  - precise knowledge of parton flux at hadron colliders essential $\longrightarrow$ Higgs or SUSY searches at LHC

Challenge

- Anomalous dimensions of structure functions at 3 loops
  - expect significant reduction of theoretical uncertainties

Strategy

- Most promising
  $\longrightarrow$ calculate Mellin moments of structure functions
  - anomalous dimensions at 1 loop and 2 loops
    Gross, Wilczek '73; Gonzalez-Arroyo, Lopez, Ynduráin '79
  - $R = \sigma_L/\sigma_T$ of DIS structure functions at 2 loops
    Kazakov, Kotikov '88

- First steps towards 3 loops
  - reproduce F_2, F_3 and F_L at 2 loops
    S.M., Vermaseren '99

OPE and Mellin moments

- Hadronic tensor $W_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}F_3(x,Q^2)$$

- Optical theorem:
  - $W_{\mu\nu} \longrightarrow$ imaginary part of Compton amplitude $T_{\mu\nu}$

- Bjorken limit: x fixed, $Q^2 \to \infty$
  - operator product expansion of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$
    Christ, Hasslacher, Mueller '72

$$T_{\mu\nu} \;=\; \mathrm{i}\int d^4z\,\mathrm{e}^{\mathrm{i}q\cdot z}\langle \mathrm{P}|\,T\left(J^\dagger_\mu(z)J_\nu(0)\right)|\mathrm{P}\rangle \;=$$

$$= \sum_{N,j}\left(\frac{1}{2x}\right)^N\left[e_{\mu\nu}\,C^N_{L,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right) + d_{\mu\nu}\,C^N_{2,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right.$$

$$\left.+\mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}C^N_{3,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right]A^j_{\mathrm{P},N}\left(\mu^2\right) \;+\; \text{higher twists}$$

  - $C^N_{2,i} \longrightarrow$ coefficient functions in Mellin space
  - $A^i_{\mathrm{P},N} \longrightarrow$ matrix elements of operators O^i of leading twist

$$O^{\alpha,\{\mu_1,\cdots,\mu_N\}} = \overline{\psi}\lambda^\alpha\gamma^{\{\mu_1}D^{\mu_2}\ldots D^{\mu_N\}}\psi, \quad \alpha = 1,2,\ldots,(n_f^2-1)$$

$$O^{\mathrm{q},\{\mu_1,\cdots,\mu_N\}} = \overline{\psi}\gamma^{\{\mu_1}D^{\mu_2}\ldots D^{\mu_N\}}\psi,$$

$$O^{\mathrm{g},\{\mu_1,\cdots,\mu_N\}} = F^{\{\nu\mu_1}D^{\mu_2}\ldots D^{\mu_{N-1}}F^{\mu_N\nu\}}$$

Upshot

- Mellin moments of F_2, F_3 and F_L $\longleftrightarrow$ parameters of OPE

$$\int_0^1 dx\, x^{N-2} F_2(x,Q^2) \;=\; \sum_{i=\mathrm{ns,q,g}} C^N_{2,i}\left(\frac{Q^2}{\mu^2},\alpha_s\right) A^i_{\mathrm{P},N}(\mu^2)$$

Task

- Determine coefficient functions $C^N_{i,j}$ and their scale dependence
  - calculable in dimensionally regularized perturbation theory

- Scale dependence $\longrightarrow$ renormalization group equation

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} - \gamma_{\mathrm{ns}}(\alpha_s,N)\right] C^N_{2,\mathrm{ns}}\left(\frac{Q^2}{\mu^2},\alpha_s\right) = 0$$

  - $\beta(\alpha_s)$ $\longrightarrow$ QCD β-function

$$\frac{d}{d\ln\mu^2}\alpha_s = \beta(\alpha_s)$$

  - $\gamma(\alpha_s,N)$ $\longrightarrow$ anomalous dimension
    renormalized operators $O^{\mathrm{bare}} \;=\; Z\; O^{\mathrm{ren}}$

$$\frac{d}{d\ln\mu^2}O^{\mathrm{ren}} = -\gamma\; O^{\mathrm{ren}} \qquad\qquad \gamma \;=\; \left(\frac{d}{d\ln\mu^2}Z\right) Z^{-1}$$

- Solution to RGE up to NNLO
  - sum of terms inside $\{\ldots\}$ scheme independent

$$C^N_{2,\mathrm{ns}}\left(\frac{Q^2}{\mu^2},\alpha_s\right) = B^{\mathrm{ns}}(\mu^2)\Bigg[1+\frac{\alpha_s(Q)}{4\pi}\left\{c^{(1)}_{2,\mathrm{q}}+\frac{\gamma^{(1),\mathrm{ns}}_{\mathrm{qq}}}{2\beta_0}-\frac{\gamma^{(0)}_{\mathrm{qq}}}{2\beta_0^2}\right\}$$

$$+\left(\frac{\alpha_s(Q)}{4\pi}\right)^2\Bigg\{c^{(2),\mathrm{ns}}_{2,\mathrm{q}}+c^{(1)}_{2,\mathrm{q}}\left(\frac{\gamma^{(1),\mathrm{ns}}_{\mathrm{qq}}}{2\beta_0}-\frac{\gamma^{(0)}_{\mathrm{qq}}}{2\beta_0^2}\right)+\frac{1}{2}\left(\frac{\gamma^{(1),\mathrm{ns}}_{\mathrm{qq}}}{2\beta_0}-\frac{\gamma^{(0)}_{\mathrm{qq}}}{2\beta_0^2}\right)^2$$

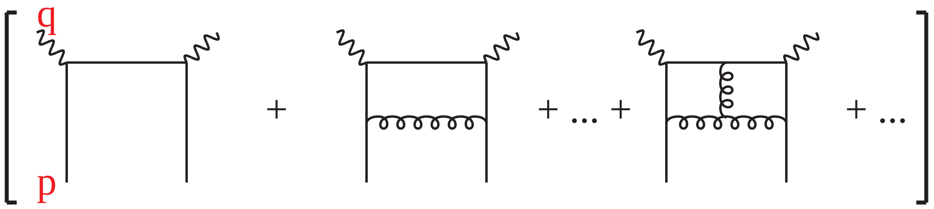
$$+\frac{\gamma^{(2),\mathrm{ns}}_{\mathrm{qq}}}{4\beta_0}-\frac{\beta_1\gamma^{(1),\mathrm{ns}}_{\mathrm{qq}}}{4\beta_0^2}-\frac{\beta_2\gamma^{(0),\mathrm{ns}}_{\mathrm{qq}}}{4\beta_0^2}+\frac{\beta_1^2\gamma^{(0),\mathrm{ns}}_{\mathrm{qq}}}{4\beta_0^3}\Bigg\}\Bigg]\left[\alpha_s(Q)\right]^{\frac{\gamma^{(0)}_{\mathrm{qq}}}{2\beta_0}}$$

  - 3-loop anomalous dimensions $\gamma^{(2)}_{\mathrm{pp}}$ unknown

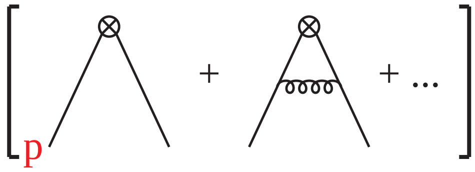
The method of projection

Gorishnii, Larin, Tkachev '83; Gorishnii, Larin '87

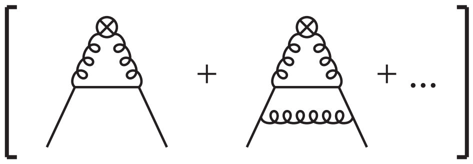
- Apply OPE to Green's functions with external partons
  - flavour singlet sector



$$= \sum_N \left(\frac{1}{2x}\right)^N \Bigg\{ \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qq}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gq}}\right)$$



$$+ \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qg}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gg}}\right)$$



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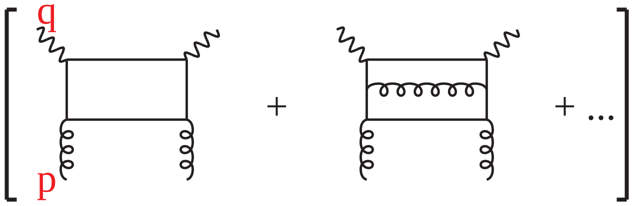

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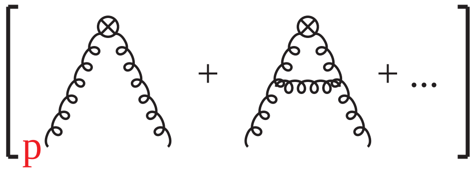
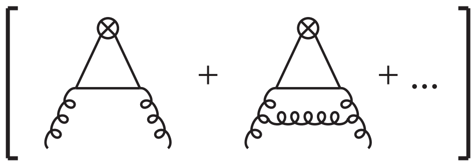


$$+ \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qq}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gq}}\right)$$

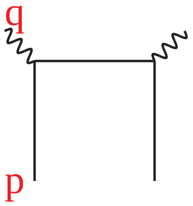


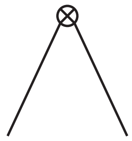
$$\Bigg\}$$

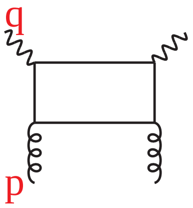
- Operator projects out N-th moment

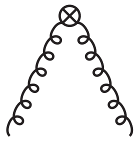
$$\mathcal{P}_N \equiv \left[\frac{q^{\{\mu_1} \cdots q^{\mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} \right] \Bigg|_{p=0}$$

- Projection $\longrightarrow$ only tree level operator matrix elements survive
  - method relies on dimensional regularization
  - operator mixing in flavour singlet sector decoupled

$\mathcal{P}_N \Big[$ 

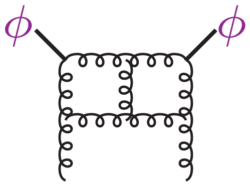
$+ \dots \Big] = \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qq}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gq}}\right)$ 

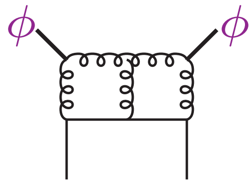
$\mathcal{P}_N \Big[$ 

$+ \dots \Big] = \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qg}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gg}}\right)$ 

- Photons only couple to quarks $\longrightarrow$ γ_{gq}, γ_{gg} only to $(l-1)$-loop

- Calculate Green's function of external scalar particle ϕ
  $\longrightarrow$ ϕ couples only to gluons
  - Feynman rules $\longrightarrow$ add term $\phi F^a_{\mu\nu} F^{\mu\nu}_a$ to QCD Lagrangian





  - OPE with same operators O^i of leading twist
    $\longrightarrow$ same renormalizations factors Z^{gq}, Z^{gg}

Reconstruction of anomalous dimensions

- Anomalous dimensions of quark operators O^{q_i}
  - split into valence and sea part

$$\gamma_{\mathrm{q}_i\mathrm{q}_j} = \gamma_{\bar{\mathrm{q}}_i\bar{\mathrm{q}}_j} = \delta_{ij}\gamma^{\mathrm{V}}_{\mathrm{qq}} + \gamma^{\mathrm{S}}_{\mathrm{qq}}$$

$$\gamma_{\mathrm{q}_i\bar{\mathrm{q}}_j} = \gamma_{\bar{\mathrm{q}}_i\mathrm{q}_j} = \delta_{ij}\gamma^{\mathrm{V}}_{\mathrm{q}\bar{\mathrm{q}}} + \gamma^{\mathrm{S}}_{\mathrm{q}\bar{\mathrm{q}}}$$

- Matrix elements of quark operators O^{q_i}
  $\longrightarrow$ non-singlet and singlet distributions $\mathrm{q}^{\pm}_{\mathrm{ns},ij}$, $\mathrm{q}^{\mathrm{V}}_{\mathrm{ns}}$ and q_{s}
  - renormalization

$$\mathrm{q}^{\pm}_{\mathrm{ns},ij} \longrightarrow \delta_{ij}\left(\gamma^{\mathrm{V}}_{\mathrm{qq}} \pm \gamma^{\mathrm{V}}_{\mathrm{q}\bar{\mathrm{q}}}\right)$$

$$\mathrm{q}^{\mathrm{V}}_{\mathrm{ns}} \longrightarrow \gamma^{\mathrm{V}}_{\mathrm{qq}} - \gamma^{\mathrm{V}}_{\mathrm{q}\bar{\mathrm{q}}} + n_f\left(\gamma^{\mathrm{S}}_{\mathrm{qq}} - \gamma^{\mathrm{S}}_{\mathrm{q}\bar{\mathrm{q}}}\right)$$

$$\mathrm{q}_{\mathrm{s}} \longrightarrow \gamma^{\mathrm{V}}_{\mathrm{qq}} + \gamma^{\mathrm{V}}_{\mathrm{q}\bar{\mathrm{q}}} + n_f\left(\gamma^{\mathrm{S}}_{\mathrm{qq}} + \gamma^{\mathrm{S}}_{\mathrm{q}\bar{\mathrm{q}}}\right)$$

- Electromagnetic interaction:
  even N of $F_2^{e\mathrm{P}} \longrightarrow$ complete flavour singlet sector
  Larin, Nogueira, van Ritbergen, Vermaseren '97
  - determine γ_{qg}, γ_{gq}, γ_{gg}, $\gamma^{\mathrm{V}}_{\mathrm{qq}} + \gamma^{\mathrm{V}}_{\mathrm{q}\bar{\mathrm{q}}}$ and $\gamma^{\mathrm{S}}_{\mathrm{qq}} + \gamma^{\mathrm{S}}_{\mathrm{q}\bar{\mathrm{q}}}$

- Electroweak interactions:
  even N of $F_2^{\nu\mathrm{P}+\overline{\nu}\mathrm{P}}$, $F_3^{\nu\mathrm{P}-\overline{\nu}\mathrm{P}}$; odd N of $F_2^{\nu\mathrm{P}-\overline{\nu}\mathrm{P}}$, $F_3^{\nu\mathrm{P}+\overline{\nu}\mathrm{P}}$
  $\longrightarrow$ complete flavour non-singlet sector
  S.M., Vermaseren '99
  - determine individually $\gamma^{\mathrm{V}}_{\mathrm{qq}}$, $\gamma^{\mathrm{V}}_{\mathrm{q}\bar{\mathrm{q}}}$, $\gamma^{\mathrm{S}}_{\mathrm{qq}}$ and $\gamma^{\mathrm{S}}_{\mathrm{q}\bar{\mathrm{q}}}$

- Important step in reconstruction
  - analytic continuation $\longrightarrow$ inverse Mellin transformation

$$S_{n_1,\ldots,n_{w+1}}(N) \longleftrightarrow H_{m_1,\ldots,m_w}(x)/(1 \pm x)$$

  - combine results for $F_2^{\nu\mathrm{P}\pm\overline{\nu}\mathrm{P}}$ and $F_3^{\nu\mathrm{P}\pm\overline{\nu}\mathrm{P}}$

- Same reconstruction for quark coefficient functions
  - $F_2^{\nu P \pm \overline{\nu} P}$, $F_3^{\nu P \pm \overline{\nu} P}$ $\longrightarrow$ four different physical processes
  - for example at 2 loops
    $\longrightarrow$ four different coefficient functions $c_{2,\mathrm{q}}^{(2),\mathrm{ns},\pm}$, $c_{3,\mathrm{q}}^{(2),\mathrm{ns},\pm}$
    S.M., Vermaseren '99

Upshot

- F_2^{eP}, $F_2^{\nu P \pm \overline{\nu} P}$ and $F_3^{\nu P \pm \overline{\nu} P}$ provide complete knowledge of
  - all anomalous dimensions $\longrightarrow$ splitting functions in x-space
  - all coefficient functions in x- or N-space

- In particular at 3 loops
  - determine individually $\gamma_{\mathrm{qq}}^{(2),\mathrm{V}}$, $\gamma_{\mathrm{q\bar{q}}}^{(2),\mathrm{V}}$, $\gamma_{\mathrm{qq}}^{(2),\mathrm{S}}$, $\gamma_{\mathrm{q\bar{q}}}^{(2),\mathrm{S}}$, $\gamma_{\mathrm{qg}}^{(2)}$, $\gamma_{\mathrm{gq}}^{(2)}$ and $\gamma_{\mathrm{gg}}^{(2)}$

Calculation up to 2 loops

The calculation

- DIS structure functions up to 2 loops
  - 425 diagrams for $F_2^{e\mathrm{P}}$
  - 360 diagrams for $F_2^{\nu\mathrm{P}\pm\overline{\nu}\mathrm{P}}$ and $F_3^{\nu\mathrm{P}\pm\overline{\nu}\mathrm{P}}$

- Optimized FORM programs
  $\longrightarrow$ several hours CPU time on Pentium 450 PC.

The tools

- QGRAF $\longrightarrow$ generation of diagrams for DIS structure functions
  Nogueira '93

- MINCER $\longrightarrow$ two-point functions up to 3 loops
  Larin, Tkachov, Vermaseren '91

- Set of very flexible FORM programs

  - SUMMER $\longrightarrow$ summation algorithm for harmonic sums
    Vermaseren '98
  - ENSUM $\longrightarrow$ recursion relations for all 2 loop topologies
    S.M., Vermaseren '99
  - MELTRAN $\longrightarrow$ inverse Mellin transformation
    Remiddi, Vermaseren '99

The checks

- Lower fixed moments calculated with gauge parameter ξ

$$\mathrm{i}\,\frac{-g^{\mu\nu}+(1-\xi)q^{\mu}q^{\nu}}{q^2-\mathrm{i}\epsilon}$$

- Some fixed moments known for each diagram
  Larin, Vermaseren '93; Larin, Nogueira, van Ritbergen, Vermaseren '97

- Agreement with:
  - 2-loop anomalous dimensions
    Floratos, Ross, Sachrajda '79; Gonzalez-Arroyo, Lopez, Ynduráin '79; Curci, Furmanski, Petronzio '80; Furmanski, Petronzio '80
  - complete 2-loop F_2, F_3 and F_L
    Zijlstra, van Neerven '92

Conclusions

Conclusions

- Powerful technology to calculate Mellin moments of higher loop corrections for DIS structure functions.

- Unpolarized DIS structure functions F_2, F_3 and F_L at 2 loops recalculated.

Outlook

- Check of polarized DIS structure function g_1 at 2 loops is under way.
  S.M., Vermaseren

- Calculation of 3-loop anomalous dimensions is definitely possible.