Differential Equations for Two-Loop Four-Point Functions

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Loops and Legs in Quantum Field Theory
Bastei / Königstein
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Introduction

Experimental precision for many $2 \to 2$ and $1 \to 3$ reactions has reached a level that demands theoretical predictions at next-to-next-to-leading order.

Examples:

- Bhabha-Scattering: $e^+e^- \to e^+e^-$
- Three Jet observables in e^+e^-
- DIS (2+1)-Jet production
- Hadron-Hadron 2-Jet and $V+1$-Jet production

Major missing ingredient for virtual corrections to these observables:

Two-loop four-point integrals with massless propagators and up to one off-shell leg

Reduction of Scalar Two-Loop Four-Point Functions

Generic structure of scalar two-loop integrals:

$$I_{t,r,s}(p_1,\dots,p_n) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{D_1^{m_1}\dots D_t^{m_t}} S_1^{n_1}\dots S_q^{n_q}$$

D_i	:	massless scalar propagators
S_i	:	scalar products involving loop momenta
t	:	number of different propagators
$r=\sum_i m_i$	:	dimension of denominator
$s=\sum_i n_i$	:	dimension of numerator

Topology of Feynman graph defined by specifying the set of different propagators

$$\{D_1,\dots,D_t\}$$

Reduction of Scalar Two-Loop Four-Point Functions

Identities:

- Integration-by-parts (IBP)
  K. Chetyrkin, F. Tkachov

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{\partial}{\partial a^\mu} \left[b^\mu f(k,l,p_i) \right] = 0$$

  with: $a^\mu = k^\mu, l^\mu$ and $b^\mu = k^\mu, l^\mu, p_i^\mu$

- Lorentz Invariance (LI)
  E. Remiddi, TG

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \delta\varepsilon^\mu_\nu \left(\sum_i p_i^\nu \frac{\partial}{\partial p_i^\mu} \right) f(k,l,p_i) = 0$$

For each two-loop four-point integral, one has 10 IBP and 3 LI identities.

Reduction of Scalar Two-Loop Four-Point Functions

The IBP and LI identities for $I_{t,r,s}$ relate:

$I_{t,r,s}$	:	the integral itself
$I_{t-1,r,s}$	:	simpler topology
$I_{t,r+1,s}, I_{t,r+1,s+1}$	:	same topology, more complicated than $I_{t,r,s}$
$I_{t,r-1,s}, I_{t,r-1,s-1}$	:	same topology, simpler than $I_{t,r,s}$

In numbers:

$\boxed{t = 7}$

different $I_{t,r,s}$

r \ s	0	1	2	3	4
7	1	2	3	4	5
8	7	14	21	28	35
9	28	56	84	112	140
10	84	168	252	336	420

accumulated equations
unknowns

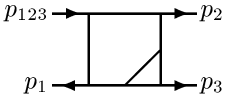
r \ s	0	1	2	3
7	13 22	39 45	78 76	130 115
8	104 106	312 213	624 354	1040 535
9	468 358	1404 717	2808 1196	4680 1795

\# equations grows faster than # unknowns

$\longrightarrow$ Reduction possible (using MAPLE and FORM)

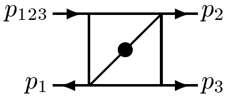
Reduction of Scalar Two-Loop Four-Point Functions

Example of a reducible integral:

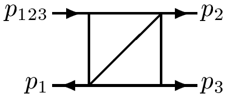


$$=$$

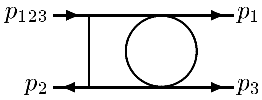
$$\frac{2(2d-9)}{(d-4)(d-6)}\,\frac{2s_{12}+s_{13}+s_{23}}{s_{13}s_{23}}$$



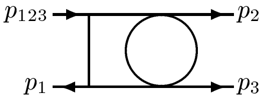
$$-\frac{3(d-4)}{d-6}\,\frac{(s_{13}+s_{23})^2}{s_{13}^2 s_{23}^2}$$



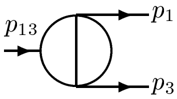
$$-\frac{6(d-3)(3d-14)}{(d-4)(d-6)}\,\frac{(s_{12}+s_{23})^2}{s_{13}^2 s_{23}^2}$$



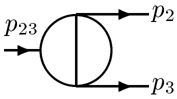
$$-\frac{6(d-3)(3d-14)}{(d-4)(d-6)}\,\frac{(s_{12}+s_{13})^2}{s_{13}^2 s_{23}^2}$$



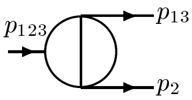
$$+\frac{3(d-3)(3d-10)(3d-14)}{2(d-4)^2(d-5)(d-6)}\,\frac{(2d-10)s_{12}+(3d-14)s_{23}}{s_{13}^2 s_{23}^2}$$



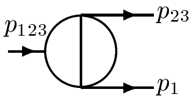
$$+\frac{3(d-3)(3d-10)(3d-14)}{2(d-4)^2(d-5)(d-6)}\,\frac{(2d-10)s_{12}+(3d-14)s_{13}}{s_{13}^2 s_{23}^2}$$



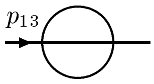
$$-\frac{3(d-3)(3d-10)}{(d-4)^2(d-6)}\,\frac{(3d-14)s_{12}+(4d-18)s_{13}-(d-4)s_{23}}{s_{13}^2 s_{23}^2}$$



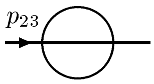
$$-\frac{3(d-3)(3d-10)}{(d-4)^2(d-6)}\,\frac{(3d-14)s_{12}-(d-4)s_{13}+(4d-18)s_{23}}{s_{13}^2 s_{23}^2}$$



$$-\frac{3(d-3)(3d-8)(3d-10)}{(d-4)^3(d-5)(d-6)}\,\frac{(d-5)(3d-14)(s_{12}+s_{13})+(d-4)^2 s_{23}}{s_{13}^3 s_{23}^2}$$



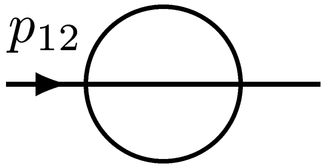
$$-\frac{3(d-3)(3d-8)(3d-10)}{(d-4)^3(d-5)(d-6)}\,\frac{(d-5)(3d-14)(s_{12}+s_{23})+(d-4)^2 s_{13}}{s_{13}^2 s_{23}^3}$$



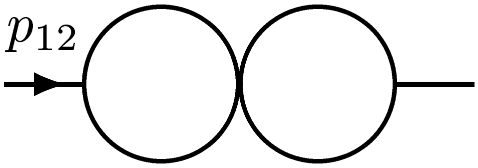
Calculation of Master Integrals

Genuine one-scale master integrals:

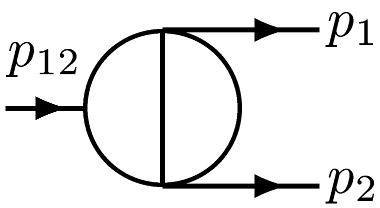
R. Gonsalves; G. Kramer, B. Lampe



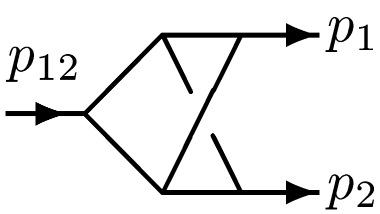
$$= \quad A_3 \left(-p_{12}^2\right)^{d-3}$$



$$= \quad A_{2,LO}^2 \left(-p_{12}^2\right)^{d-4}$$



$$= \quad A_4 \left(-p_{12}^2\right)^{d-4}$$



$$= \quad A_6 \left(-p_{12}^2\right)^{d-6}$$

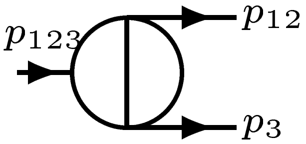
Calculation of Master Integrals

E. Remiddi

Multi-scale master integrals fulfil inhomogeneous differential equations in their external invariants.

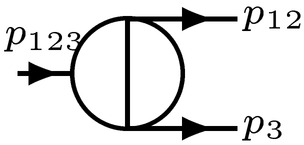
For example:

$$s_{123}\frac{\partial}{\partial s_{123}}$$

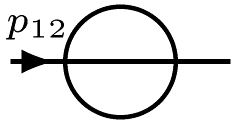


$$=$$

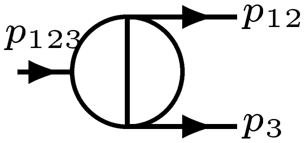
$$+\frac{d-4}{2}\,\frac{2s_{123}-s_{12}}{s_{123}-s_{12}}$$



$$-\frac{3d-8}{2}\,\frac{1}{s_{123}-s_{12}}$$

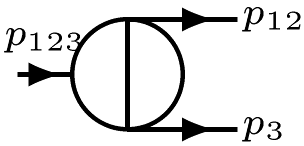


$$s_{12}\frac{\partial}{\partial s_{12}}$$

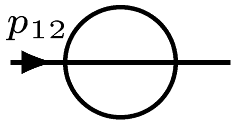


$$=$$

$$-\frac{d-4}{2}\,\frac{s_{12}}{s_{123}-s_{12}}$$



$$+\frac{3d-8}{2}\,\frac{1}{s_{123}-s_{12}}$$

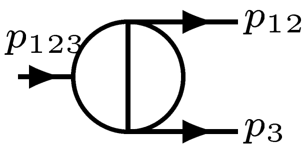


Calculation of Master Integrals

E. Remiddi, TG

- Boundary conditions for differential equations correspond to simpler integrals with fewer scales
- Solution of differential equations by integration and matching of boundary conditions

For example:



$$= \quad A_4 \left(s_{12} - s_{123}\right)^{\frac{d}{2}-2} \left(-s_{123}\right)^{\frac{d}{2}-2}$$

$$- \frac{3d-8}{2(d-3)} A_3 \frac{\left(-s_{12}\right)^{d-3}}{-s_{123}} \, {}_2F_1\left(\frac{d}{2}-1, 1; d-2; \frac{s_{12}}{s_{123}}\right)$$

Summary and Outlook

Reduction of multi-leg integrals to a small set of master integrals using Integration-by-parts method and Lorentz-invariance

Calculation of master integrals using differential equations in external invariants

Applications:

- Limit $p_{123}^2 \to 0$: massless two-loop on-shell $2 \to 2$ amplitudes
  C. Anastasiou, C. Oleari, E. Remiddi, J.B. Tausk, TG
  $\longrightarrow$ Talks of Bas Tausk, Nigel Glover
- Master integrals for massless two-loop four-point functions with one off-shell leg:
  $e^+e^- \to 3$ Jets, $ep \to (2+1)$ Jets,
  $pp \to (W^\pm, Z^0, \gamma^*) + 1$ Jet
  (work in progress)