Factorization Approach for Top Mass Reconstruction in the Continuum

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Using effective theories for jets and heavy quarks it is possible to prove that the double differential top-antitop invariant mass distribution for the process $e^+e^- \to t\bar{t}$ in the resonance region for c.m. energies Q much larger than the top mass can factorized into perturbatively computable hard coefficients and jet functions and a non-perturbative soft function. For invariant mass prescriptions based on hemispheres defined with respect to the thrust axis the soft function can be extracted from massless jet event shape distributions. This approach allows in principle for top mass determinations without uncertainties from hadronization using the reconstruction method and to quantify the top mass scheme dependence of the measured top quark mass value.

1 Introduction

Precise measurements of the top quark mass are among the most important (standard) tasks of the ILC project as the top quark mass affects a number of interesting observables either directly or indirectly through quantum effects. To be useful such top mass measurements have to have small uncertainties, but also need to provide information to which mass scheme the measured number refers to. Both aims can be achieved from a threshold scan of the cross section $\sigma(e^+e^- \to t\bar{t})$ for $\sqrt{s} \approx 2m_t$, from which one expects measurements of the threshold masses, such as the 1S mass, with uncertainties of about 100 MeV [2, 3, 4, 5, 6]. Another method is based on mass reconstruction which relies on the idea that the peak of the invariant mass distribution of the top decay products is related to the top quark mass. This method can be applied at any c.m. energy and might also yield uncertainties well below 1 GeV [7]. However, until recently it was unknown for which mass scheme such measurements can be carried out with small theoretical uncertainties. This is because the naive relation between the observable peak of the invariant mass distribution and the perturbative top quark propagator pole is affected by hard (i.e. computable) as well as soft (i.e. non-perturbative) QCD effects and the present MC tools do not contain the required information in a systematic form. Obviously the top mass measurements at the LHC [9] suffer from the same problem, but the associated theoretical systematic uncertainty might be considerably larger than at the ILC.

2 Factorization Theorem

In Ref. [8] a theoretical formalism was presented which remedies this situation, as a first step, for the Linear Collider framework, where one does not need to account for QCD radiation

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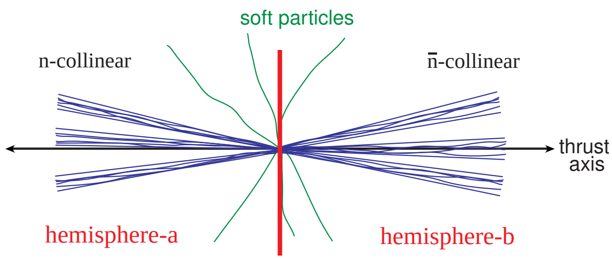

Figure 1: Six jet event initiated by a top quark pair, $t\bar{t} \to bW\bar{b}W \to bqq'\bar{b}qq'$. The plane separating the two hemispheres is perpendicular to the thrust axis and intersects the thrust axis at the interaction point. The total invariant mass inside each hemisphere is measured. Our analysis applies equally well to the lepton+jets and the dilepton channels (not shown).

arising from the initial state. Assuming a c.m. energy $Q \gg m_t$, m_t being the top quark mass, one can employ the hierarchy of scales

$$Q \gg m_t \gg \Gamma_t > \Lambda_{\rm QCD} \tag{1}$$

to establish a factorization theorem for the doubly differential top-antitop invariant mass distribution in the peak region around the top resonance:

$$\frac{d^2\sigma}{dM_t^2\, dM_{\bar{t}}^2}\,, \qquad\qquad M_{t,\bar{t}}^2 - m^2 \sim m\,\Gamma \ll m^2\,. \tag{2}$$

The invariant masses $M_t^2 = (\sum_{i\in X_t} p_i^\mu)^2$, $M_{\bar{t}}^2 = (\sum_{i\in X_{\bar{t}}} p_i^\mu)^2$ depend on a prescription $X_{t,\bar{t}}$ which associates final state momenta p_i^μ to top and antitop invariant masses, respectively. For invariant masses in the resonance region the events are characterized by energy deposits predominantly contained in two back-to-back regions with opening angles m_t/Q associated with the energetic jets or leptons from the top decay plus collinear radiation, and by additional soft radiation populating the regions between the jets, see Fig. 1. We assume that the prescriptions $X_{t,\bar{t}}$ assign all soft radiation to either M_t^2 or $M_{\bar{t}}^2$ where the probability of radiation being assigned to X_t or $X_{\bar{t}}$ increases to unity when it approaches the top or antitop direction. The result for the double differential cross-section in the peak region at all orders in α_s and to leading order in the power expansion in $m_t\alpha_s/Q$, m_t^2/Q^2, Γ_t/m_t and $M_{t,\bar{t}} - m_t$ is given by [8]

$$\begin{aligned}\frac{d\sigma}{dM_t^2\, dM_{\bar{t}}^2} &= \sigma_0\, H_Q(Q,\mu_m) H_m\Big(m_J, \frac{Q}{m_J}, \mu_m, \mu\Big) \qquad\qquad \left[\hat{s}_{t,\bar{t}} = \frac{M_t^2 - m_J^2}{m_J}\right] \\ &\times \int d\ell^+ d\ell^- B_+\Big(\hat{s}_t - \frac{Q\ell^+}{m_J}, \Gamma_t, \mu\Big)\, B_-\Big(\hat{s}_{\bar{t}} - \frac{Q\ell^-}{m_J}, \Gamma_t, \mu\Big) S(\ell^+, \ell^-, \mu)\,.\end{aligned} \tag{3}$$

In Eq. (3) the normalization factor σ_0 is the total Born-level cross-section, the H_Q and H_m are perturbative coefficients describing hard effects at the scales Q and m_J, $B_\pm$ are perturbative jet functions that describe the evolution and decay of the the top and antitop close to the mass shell, and S is a nonperturbative soft function describing the soft radiation between the jets. The result was derived using the hierarchy of scales (1), matching QCD onto

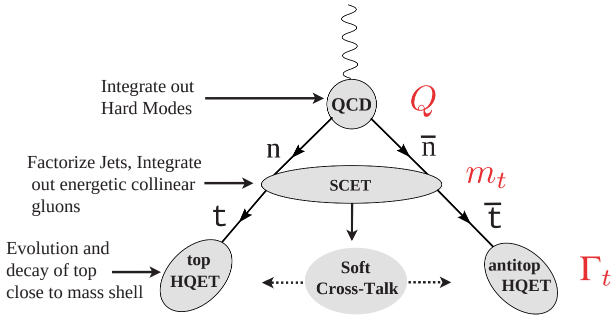

Figure 2: Sequence of effective field theories used to compute the invariant mass distribution.

(Soft-Collinear Effective Theory) SCET [10] at the scale $\mu = Q$, which in turn is matched onto (Heavy Quark Effective Theory) HQET [11] at a scale μ_m of order m_t generalized for unstable particle effects associated to the large top width Γ_t [12]. An illustration of this scheme is shown in Fig. 2. For details on the (admittedly non-trivial) derivation and on technical aspects of the factorization theorem we refer to Ref. [8]. In the following we will discuss the important ingredients of the factorization theorem and their physical interpretation and show what we can learn from them concerning the measurements of the top quark mass from the reconstruction method.

3 Jet Functions and Short-Distance Top Mass

The coefficients H_Q and H_m in Eq. (3) arise from matching and running in SCET and HQET down to the low energy scale μ where one evaluates the jet functions $B_\pm$ and the soft function S. These hard coefficients only affect the overall normalization of the invariant mass distribution and we will therefore not talk about them here. So let us concentrate on the jet and the soft functions, which determine the shape of the distribution and the location of the resonance peak. The jet functions describe the perturbative contributions of the shape of the invariant mass distribution and are defined by the imaginary part of a T-product vacuum matrix element. For the top quark it is

$$B_+(\hat{s}_t, \Gamma_t, \mu) \;=\; \mathrm{Im}\left[\frac{-i}{4\pi N_c m_J}\int d^4x\, e^{ir\cdot x}\,\langle 0|T\{\bar{h}_{v_+}(0)W_n(0)W_n^\dagger(x)h_{v_+}(x)\}|0\rangle\right], \quad (4)$$

where v_+ is the top four velocity ($v_+^2 = 1$) and $\hat{s}_t = 2v_+.r$ and h_{v_+} is the (HQET) heavy top quark field. The vacuum matrix element also contains Wilson lines of the form

$$W_n^\dagger(x) = \mathrm{P}\,\exp\left(ig\int_0^\infty ds\,\bar{n}\cdot A_+(\bar{n}s+x)\right), \quad W_n(x) = \overline{\mathrm{P}}\,\exp\left(-ig\int_0^\infty ds\,\bar{n}\cdot A_+(\bar{n}s+x)\right),$$

where $\bar{n}$ is a light-like four vector pointing in the antitop direction and A_+ is field describing a gluon that is collinear to the quark. Up to the Wilson lines the vacuum matrix element is in fact a heavy quark propagator and, indeed, at tree-level it is just

$$B_\pm^{\mathrm{tree}}(\hat{s}, \Gamma_t) = \mathrm{Im}\left[\frac{-1}{\pi m_J}\,\frac{1}{\hat{s} - 2\delta m + i\Gamma_t}\right] = \frac{1}{\pi m_J}\,\frac{\Gamma_t}{(\hat{s} - 2\delta m)^2 + \Gamma_t^2}\,,$$

which is the imaginary part of the heavy quark propagator supplemented by a constant width term and describing a Breit-Wigner distribution having a width Γ_t. The residual mass term δm becomes relevant at higher orders and controls the mass scheme that is used. For the pole mass scheme $\delta m = 0$ to all order in α_s. It is the width term (which we can approximate as a constant since we are interested in the resonance region) that allows us to use perturbation theory for computing the jet function. To understand the role of the Wilson lines recall that the two-point function of simple heavy quark fields, evaluated off-shell, is not gauge invariant, a fact that becomes e.g. apparent from the gauge parameter dependence of the perturbative corrections. The jet functions $B_\pm$, however, are gauge-invariant due to the Wilson lines and well-defined *physical* objects. Physically the Wilson lines describe gluons radiated from the antitop (moving along the four vector $\bar{n}$) that are collinear to the top quark (moving along v_+), and it this additional radiation that renders the jet function gauge-invariant and physically meaningful. In momentum space the Wilson lines lead to additional Feynman diagrams having $1/(n \cdot k \pm i0)$ eikonal propagators.

Having defined the jet function it is now straightforward to address the question which mass scheme one might employ to have a good perturbative behavior of the jet function. At one-loop [8] one finds that the peak position is located at $\hat{s}_{\rm peak} = 2\delta m - C_F \alpha_s(\mu)/2\Gamma_t[\ln(\frac{\mu}{\Gamma_t}) + \frac{3}{2}]$. Recalling also that the pole mass contains a nasty $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon, it therefore natural to use a mass scheme different from the pole mass that is renormalon free and absorbs at least the major part of the higher order corrections to the peak position such that the resulting series is convergent. The definition of such a scheme is obviously not unique and can also be defined from moments of the distribution [13]. Generically we call such a mass a "jet mass" m_J and its perturbative relation to the pole mass reads

$$m_J = m_t^{\rm pole} - \delta m \,, \tag{5}$$

where the HQET power counting requires that $\delta m \sim \alpha_s \Gamma$ in the resonance region. Using this relation one can relate the jet mass to other mass schemes. The jet mass m_J has already been used in the formulae shown before. From this examination we see that top mass one can measure from reconstruction is a jet mass. For sure, one cannot measure the $\overline{\rm MS}$ mass from reconstruction because it has $\delta m \sim \alpha_s m_t \gg \Gamma_t$ and would invalidate the HQET power counting.

4 Soft Function

The soft function $S(\ell^+, \ell^-, \mu)$ describes the non-perturbative contributions of the invariant mass distribution in the resonance region. Its definition depends on the details of the prescription how soft radiation is associated to M_t and $M_{\bar{t}}$. One possible prescription is using a hemisphere mass definition, where X_t and $X_{\bar{t}}$ contain everything to the left or right of the plane perpendicular to the thrust axis of each event, see Fig. 1. It is easy to understand that such a (and any other) prescription is leading to a non-perturbative soft function since one cannot compute perturbatively how the soft particles are distributed around the hemisphere boundary. Other prescriptions are possible as long as they do not associate soft radiation going in the top direction to the antitop and vice-versa. This is in contrast to the jet functions which, according to the condition on X_t and $X_{\bar{t}}$ stated in Sec. 2 are prescription-independent since they describe energetic jets within a small cone with opening angle m_t/Q around the top direction. At leading order in the power counting the allowed prescriptions do not affect these energetic jets. For the hemisphere prescription the soft function is defined by the vacuum matrix element

$$S_{\rm hemi}(\ell^+,\ell^-) = \frac{1}{N_c}\sum_{X_s}\delta(\ell^+ - k_s^{+a})\delta(\ell^- - k_s^{-b})\langle 0|(\overline{Y}_{\bar n})^{cd}\,(Y_n)^{ce}(0)|X_s\rangle\langle X_s|(Y_n^\dagger)^{ef}\,(\overline{Y}_{\bar n}^\dagger)^{df}(0)|0\rangle\,,$$

where c, d, e, f are color indices and the $Y's$ are Wilson lines with soft gluons of the form

$$Y_n(x) = \overline{\rm P}\,\exp\Big(-ig\int_0^\infty ds\, n\cdot A_s(ns+x)\Big)\,,\qquad Y_n^\dagger(x) = {\rm P}\,\exp\Big(ig\int_0^\infty ds\, n\cdot A_s(ns+x)\Big)\,,$$
$$\overline{Y_{\bar n}}^\dagger(x) = {\rm P}\,\exp\Big(ig\int_0^\infty ds\,\bar n\cdot \overline{A}_s(\bar ns+x)\Big)\,,\quad \overline{Y_{\bar n}}(x) = \overline{\rm P}\,\exp\Big(-ig\int_0^\infty ds\,\bar n\cdot \overline{A}_s(\bar ns+x)\Big)\,.(6)$$

The k_s^{+a} and k_s^{-b} are operators that pick, according to the hemisphere mass prescription, the total + and − light-cone momentum of the gluons that are in hemisphere a and b, respectively, see Fig. 1. These Wilson lines describe soft radiation off the top and antitop quark and also render the soft function gauge-invariant.

The factorization theorem (3) shows that the soft function needs to be convoluted with the jet functions. This can be understood physically, since the way how the soft radiation is associated to M_t and $M_{\bar t}$ has to affect the observed invariant mass distribution. Field theoretically this convolution arises from the fact that the small components of light-cone momenta in the top and antitop jets fluctuate at the same length scales as the soft momenta described by the soft function. At this point it is also useful to note that S is a renormalized object and that its renormalization group evolution can be computed in perturbation theory. Nevertheless the actual form of the soft function (i.e. the initial condition for the soft function evolution at a low energy scale) is not computable with perturbative methods. So in practice the soft function needs to be modeled and eventually fixed by experimental data, similar to parton-distribution functions. How a soft model function can be constructed incorporating consistently the required higher order perturbative information has been discussed in Ref. [14].

Given that the soft function is nonperturbative and affects the invariant mass distribution at leading order, one might ask what one has gained from predicting the invariant mass distribution based on (3) and concerning a precise measurement of the top mass from the mass $M_{t,\bar t}$ where the resonance is located. The crucial aspect is that the soft function is universal and appears also in the factorization theorem for event shape distributions for jets originating from massless quarks [15] in the dijet region, where the thrust $T \approx 1$ [16]. This is related to the fact that the soft Y Wilson lines that arise from massless and from massive quark lines are identical. So our factorization theorem for the top invariant mass distribution in the resonance region becomes predictive after having determined a soft function from event shape distributions from e^+e^- data, which are already available from LEP [17]. Such a determination of the soft function was carried out by Korchemsky and Tafat in Ref. [18].

5 Numerical Analysis at LO

Using the factorization theorem it is straightforward to carry out a simple LO analysis using the tree-level result for the jet functions (i.e. one can set $\delta m = 0$) and the soft model

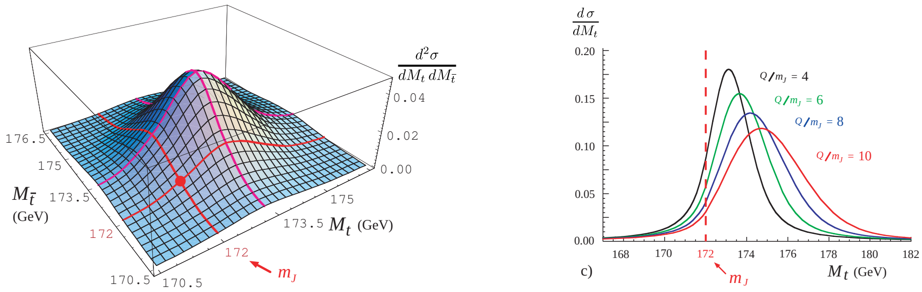

Figure 3: (a) Plot of the double differential hemisphere invariant mass cross-section $d^2\sigma/dM_t dM_{\bar{t}}$ in units of $4\sigma_0/\Gamma_t^2$ for $m_J = 172$, $Q = 4.33m_J$ and $\Gamma_t = 1.43$ GeV. (b) Dependence of the single differential invariant mass distribution as described in the text on the c.m. energy Q with the same normalization.

function determined by Korchemsky and Tafat:

$$S^{\rm M1}_{\rm hemi}(\ell^+,\ell^-) = \theta(\ell^+)\theta(\ell^-)\frac{\mathcal{N}(a,b)}{\Lambda^2}\Big(\frac{\ell^+\ell^-}{\Lambda^2}\Big)^{a-1}\exp\Big(\frac{-(\ell^+)^2-(\ell^-)^2-2b\ell^+\ell^-}{\Lambda^2}\Big)\,, \tag{7}$$

where $\mathcal{N}$ is a normalization factor. From fits to e^+e^- heavy jet mass and thrust LEP data they obtained

$$a = 2\,, \qquad b = -0.4\,, \qquad \Lambda = 0.55\,\text{GeV}\,, \tag{8}$$

which we adopt in the following. The analysis illustrates a number of important features related to how the predictions by the factorization theorem depend on the c.m. energy Q.

In Fig. 3a the double differential invariant mass distribution is displayed for the input values $m_J = 172$, $Q = 4.33m_J$ and $\Gamma_t = 1.43$ GeV. The conspicuous feature of the predicted distribution is that the observable resonance peak it shifted toward a higher value by about 1.5 GeV. This feature is one of the important properties of a invariant mass prescription that assigns all soft radiation to the masses. In the factorization theorem it arises from the Q/m factor involved in the convolution over the variables $\ell^\pm$. Intuitively it can be easily understood from the fact that the total invariant mass of a fast moving particle with mass m plus a soft momentum increases linearly with the soft momentum and the boost factor of the massive particle. This feature is also visible in Fig. 3b where the single differential invariant mass distribution

$$\frac{d\sigma}{dM_t} = \frac{2}{\Gamma}\int_{M_{\rm lower}}^{M_{\rm upper}} dM_{\bar t}\;\frac{d^2\sigma}{dM_t dM_{\bar t}}, \tag{9}$$

is plotted. Here the integration interval $[M_{\rm lower}, M_{\rm upper}]$ is twice the size of the measured peak mean half width and centered at the peak mass. For the single differential distribution one can relate the peak location approximately to the first moment of the soft function by

$$M_t^{\rm peak} \simeq m_J + \frac{Q}{2m_J}\,S^{[1,0]}_{\rm hemi}. \tag{10}$$

Interestingly this relation can also be used for fixed Q/m for invariant mass prescriptions that differ in the treatment of the soft radiation and lead to different first moments of the soft function. When extrapolated to zero moment linearly one can obtain an estimate for the jet mass.

We also find that the distribution gets wider with Q/m. This is again a consequence of the Q/m factor occurring in the convolution over $\ell^{\pm}$ in the factorization theorem since for increasing Q the jet function gets effectively smeared over a wider distribution. The shift of the peak position as well as the widening of the invariant mass distribution have been observed in simulation studies at the ILC [7] and the LHC for large p_T events [9] and can now be better quantified using the factorization theorem.

6 Acknowledgments

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